Post-Quantum Cryptography

Prof. Claude Crépeau
McGill University

Why post-Quantum ?

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- RSA

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- DSS
- Paillier

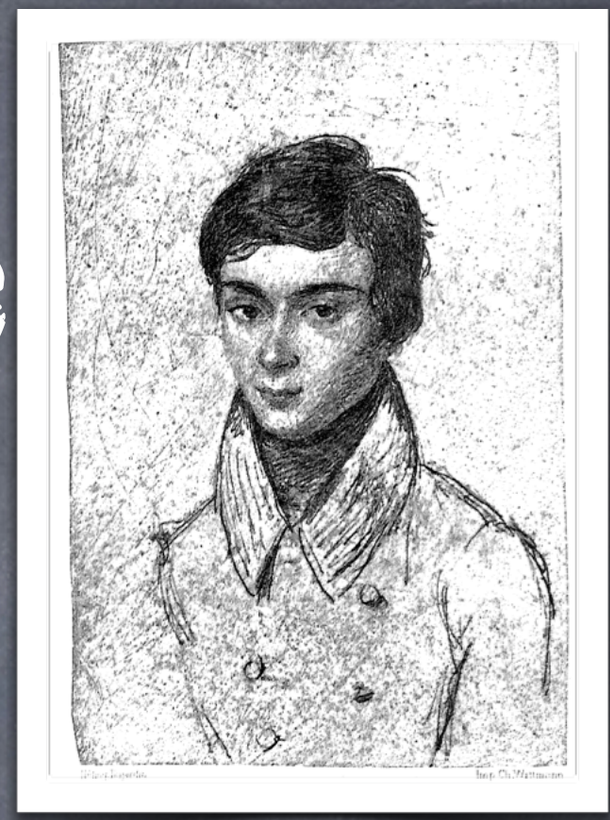
Post-Quantum Cryptography

- Finite Fields based cryptography
  - Codes
  - Multi-variate Polynomials
- Integers based cryptography
  - Approximate Integer GCD
  - Lattices

Finite Fields based cryptography

PART 1

Finite Fields



- $(F,+,x)$
- $+$: 0, $-a$, commutativity, associativity
- x : 1, a^{-1}, commutativity, associativity
- distributivity of x over $+$.
- $\mathbb{F}_p = (\{0,1,2,\ldots,p-1\}, + \bmod p, x \bmod p)$ for primes p
- $\mathbb{F}_q = (\{0,1,2,\ldots,p-1\}^m, +^m \bmod p, x \bmod P \pmod p)$
  for prime powers $q=p^m$, where P is an irreducible
  polynomial $\pmod p$ of degree $m-1$

1.5 Prime numbers

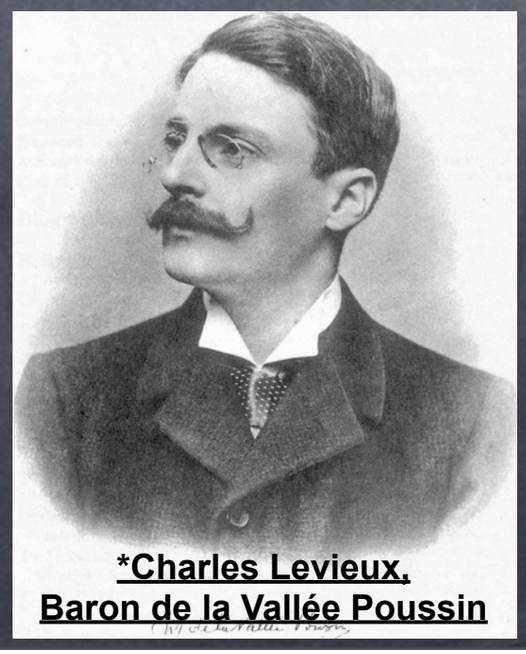
If we want a random prime (Maple `rand, isprime`) of a given size, we use the following theorem to estimate the number of integers we must try before finding a prime. Let $\pi(n) = \#\{a : 0 < a \leq n \text{ and } a \text{ is prime}\}$.

Theorem 1.9* $\lim_{n\to\infty} \dfrac{\pi(n)\log n}{n} = 1$

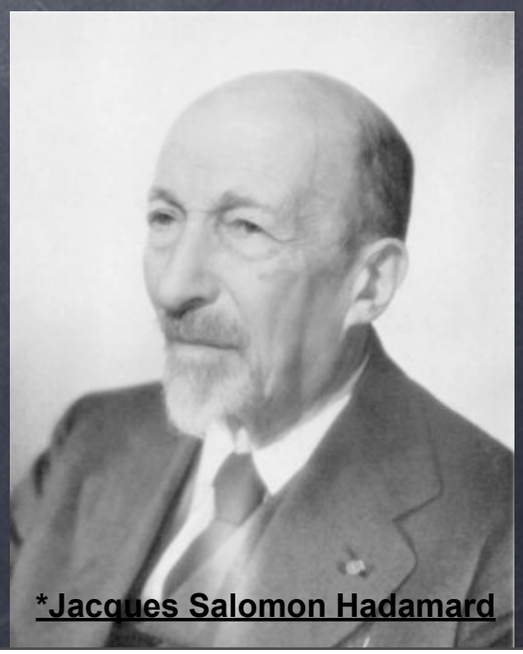
To decide whether a number n is prime or not we rely on Miller-Rabin's probabilistic algorithm. This algorithm introduces the notion of "pseudo-primality" base a. Miller defined this test as an extension of Fermat's test. If the Extended Riemann Hypothesis is true than it is sufficient to use the test with small values of a to decide whether a number n is prime or composite. However the ERH is not proven and we use the test in a probabilistic fashion as suggested by Rabin.

Gary L. **Miller**

Michael O. **Rabin**



*Charles Levieux, Baron de la Vallée Poussin



*Jacques Salomon Hadamard

Pierre de **Fermat**

1.4.2 Fermat-Euler

Theorem 1.3 (Fermat) *Let p be a prime number and a be an integer not a multiple of p, then*

$$a^{p-1} \equiv 1 \pmod{p}.$$

Theorem 1.4 (Euler) *Let p be a prime number and a be an integer, then*

$$a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \pmod{p}.$$

Theorem 1.5 (Euler) *Let n be an integer and a another integer such that $\gcd(a, n) = 1$, then*

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Algorithm 1.5 ($Pseudo(a, n)$)

1: **IF** $\gcd(a, n) \neq 1$ **THEN RETURN** *"composite"*,

2: *Let t be an odd number and s a positive integer such that $n - 1 = t2^s$*

3: $x \leftarrow a^t \bmod n$, $y \leftarrow n - 1$,

4: **FOR** $i \leftarrow 0$ **TO** s

5: **IF** $x = 1$ **AND** $y = n - 1$ **THEN RETURN** *"pseudo"*

6: $y \leftarrow x$, $x \leftarrow x^2 \bmod n$,

7: **ENDFOR**

8: **RETURN** *"composite"*.

It is easy to show that if n is prime, then $Pseudo(a, n)$ returns "pseudo" for all a, $0 < a < n$. Rabin showed that if n is composite, then $pseudo(a, n)$ returns "composite" for at least $3n/4$ of the values of a, $0 < a < n$.

Theorem 1.10

$$\#\{a : Pseudo(a, n) = \text{"pseudo"}\} \begin{cases} = \phi(n) & = n - 1 & \text{if } n \text{ is prime} \\ \leq \phi(n)/4 & \leq (n-1)/4 & \text{if } n \text{ is composite.} \end{cases}$$

2.2.1 Irreducible Polynomials

A polynomial $g(x)$ is *irreducible* (Maple `irreduc`) if it is not the product of two polynomials $h(x), k(x)$ of lower degrees. We use the following theorem to find irreducible polynomials.

Theorem 2.6 *Let $l_1, l_2, ..., l_k$ be the prime factors of n and $m_i = n/l_i$ for $1 \leq i \leq k$. A polynomial $g(x)$ of degree n is* irreducible *over $\mathcal{F}_p$ iff*

- $g(x)|x^{p^n} - x$
- $gcd(g(x), x^{p^{m_i}} - x) = 1 \; for \; 1 \leq i \leq k$

Algorithm 2.3 (Rabin $Irr(p, n)$)

1: *let* $l_1, l_2, ..., l_k$ *be the prime factors of* n *and* $m_i = n/l_i$ *for* $1 \leq i \leq k$,

2: REPEAT

3: *pick a random polynomial* $h(x)$ *of degree* $n-1$ *over* $\mathcal{F}_p$, *and set* $g(x) \leftarrow x^n + h(x)$,

4: UNTIL $x^{p^n} \bmod g(x) = x$ *and* $gcd(g(x), x^{p^{m_i}} \bmod g(x) - x) = 1$ *for* $1 \leq i \leq k$,

5: RETURN g.

We use the following theorem to estimate the number of polynomials we have to try on average before finding one that is irreducible.

Theorem 2.7 *Let* $m(n)$ *be the number of irreducible polynomials* $g(x)$ *of degree* n *of the form* $g(x) = x^n + h(x)$ *where* $h(x)$ *is of degree* $n-1$. *We have*

$$\frac{p^n}{2n} \leq \frac{p^n - p^{n/2} \log n}{n} \leq m(n) \leq \frac{p^n}{n}.$$

2.1.1 Primitive Elements

In all finite fields $\mathcal{F}_q$ (and some groups in general) there exists a *primitive element*, that is an element g of the field such that $g^1, g^2, ..., g^{q-1}$ enumerate all of the $q-1$ non-zero elements of the field. We use the following theorem to find a primitive element over $\mathcal{F}_q$.

Theorem 2.1 *Let* $l_1, l_2, ..., l_k$ *be the prime factors of* $q-1$ *and* $m_i = (q-1)/l_i$ *for* $1 \leq i \leq k$. *An element* g *is* primitive *over* $\mathcal{F}_q$ *if and only if*

- $g^{q-1} = 1$
- $g^{m_i} \neq 1$ *for* $1 \leq i \leq k$

Algorithm 2.1 (*Primitive*(q))

1: *Let* $l_1, l_2, ..., l_k$ *be the prime factors of* $q-1$ *and* $m_i = \frac{q-1}{l_i}$ *for* $1 \leq i \leq k$,

2: **REPEAT**

3: *pick a random non-zero element* g *of* $\mathcal{F}_q$,

4: **UNTIL** $g^{m_i} \neq 1$ *for* $1 \leq i \leq k$,

5: **RETURN** g.

(Maple `primroot, G[PrimitiveElement]`)

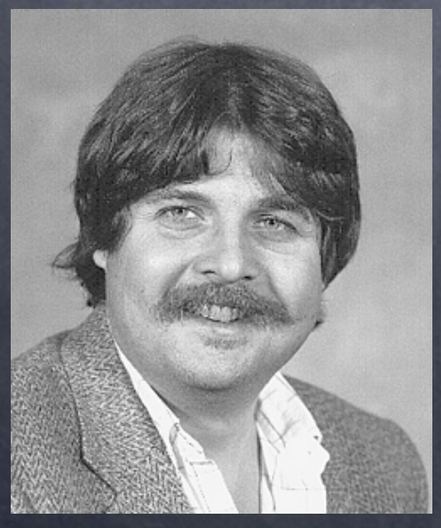
We use the following theorems to estimate the number of field elements we must try in order to find a random primitive element.

Theorem 2.2 $\#\{g : g \text{ is a primitive element of } \mathcal{F}_q\} = \phi(q-1)$.

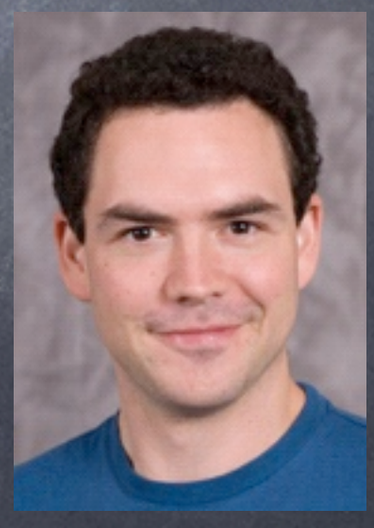
Theorem 2.3 $\liminf_{n\to\infty} \dfrac{\phi(n)\log\log n}{n} = e^{-\gamma} \approx 0.5614594836$

Example: 2 is a primitive element of $\mathcal{F}_5$ since $\{2, 2^2, 2^3, 2^4\} = \{2, 4, 3, 1\}$.

Factoring $q-1$... The only efficient way we know to finding a primitive element in fields $\mathcal{F}_q$ is when the factorization of $q-1$ is known. In general, it may be difficult to factor $q-1$. However, if we are after a large field with a random number of elements, Eric Bach has devised an efficient probabilistic algorithm to generate random integers of a given size with known factorization. Recently, Adam Kalai has invented a somewhat slower algorithm that is much simpler. Suppose we randomly select r with its factorization using Bach's or Kalai's algorithm. We may check whether $r+1$ is a prime or a prime power. In this case a finite field of $r+1$ elements is obtained and a primitive element may be computed.

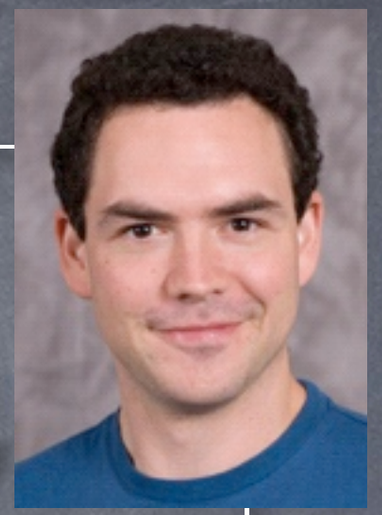


Eric Bach



Adam Kalai

http://homes.cerias.purdue.edu/~ssw/cun/index.html



Algorithm 2.2 (Kalai $randfact(n)$ **)**

1: *Generate a sequence* $n = s_0 \geq s_1 \geq s_2 \geq ... \geq s_\ell = 1$ *by picking* $s_{i+1} \in_R \{1, 2, ..., s_i\}$, *until reaching* $s_\ell = 1$.

2: *Let* r *be the product of the prime* s_i*'s,* $1 \leq i \leq \ell$.

3: IF $r \leq n$ **THEN** *with probability* r/n **RETURN** $(r, \{prime\ s_i\text{'s}\})$.

4: *Otherwise,* **RESTART**.

Theorem 2.4 *The probability of producing* r *at step 2 is* M_n/r, *where* $M_n = \prod_{p \leq n}(1 - 1/p)$.

Thus by outputting r with probability r/n in step 3, each possible value is generated with equal probability $\frac{M_n}{r}\frac{r}{n} = \frac{M_n}{n}$. The overall probability that some small enough r is produced and chosen in step 3 is $\sum_{1 \leq r \leq n} \frac{M_n}{n} = M_n$.

Theorem 2.5 $\lim_{n \to \infty} M_n \log n = e^{-\gamma} \approx 0.5614594836$

Code based cryptography

§

basic Coding theory

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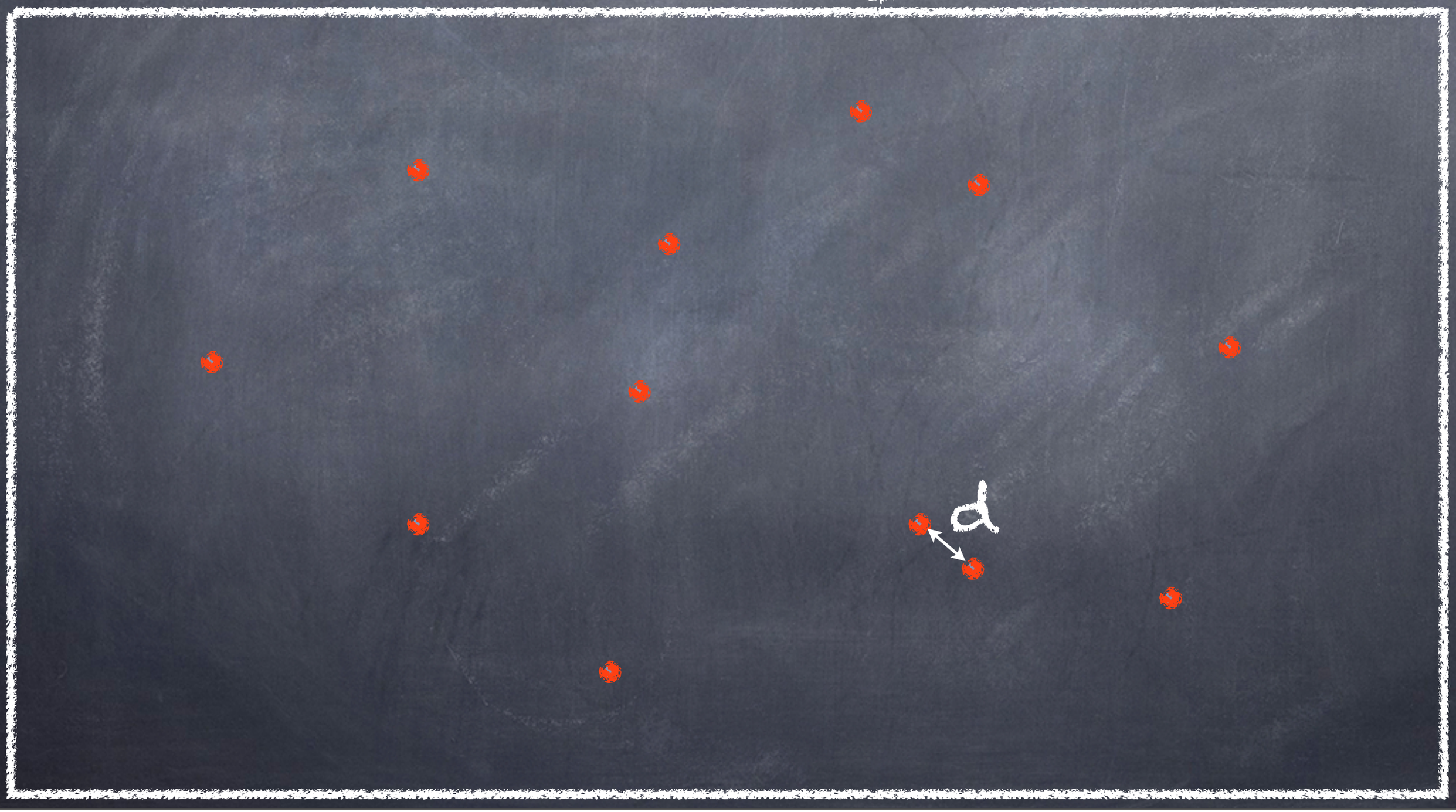
basic Coding theory

$\mathbb{F}_q^N$



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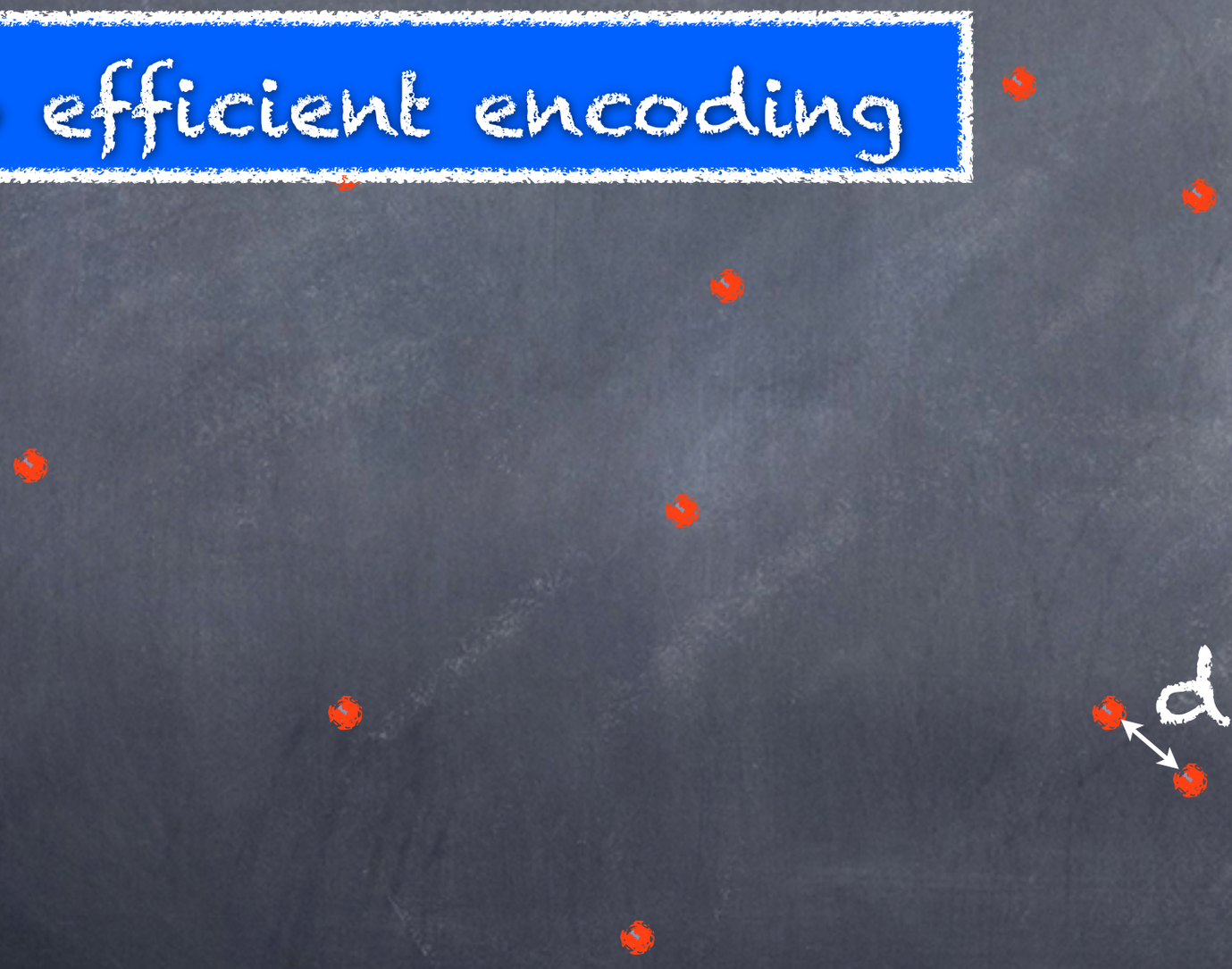
- No efficient encoding



basic Coding theory

$\mathbb{F}_q^N$

- No efficient encoding
- No efficient decoding

basic Coding theory

$\mathbb{F}_q^N$

- No efficient encoding
- No efficient decoding

- No efficient error detection

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- No efficient encoding
- No efficient decoding
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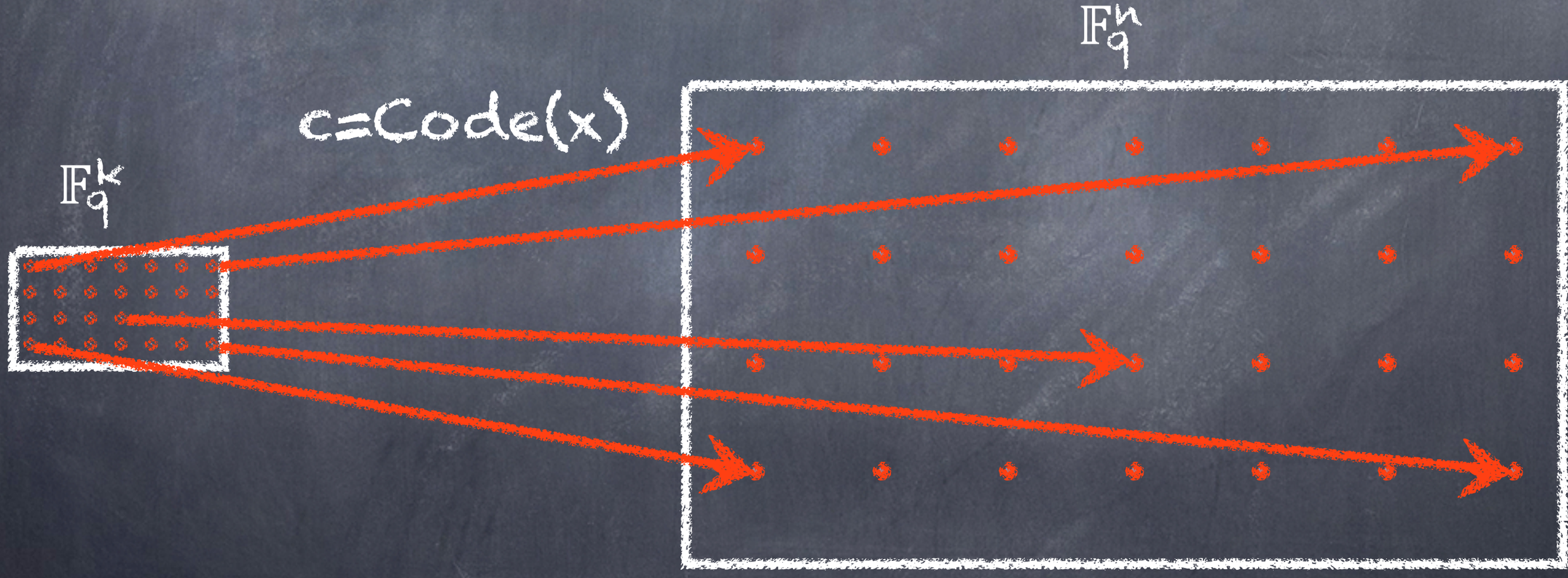
- No efficient error correction

coding vs decoding



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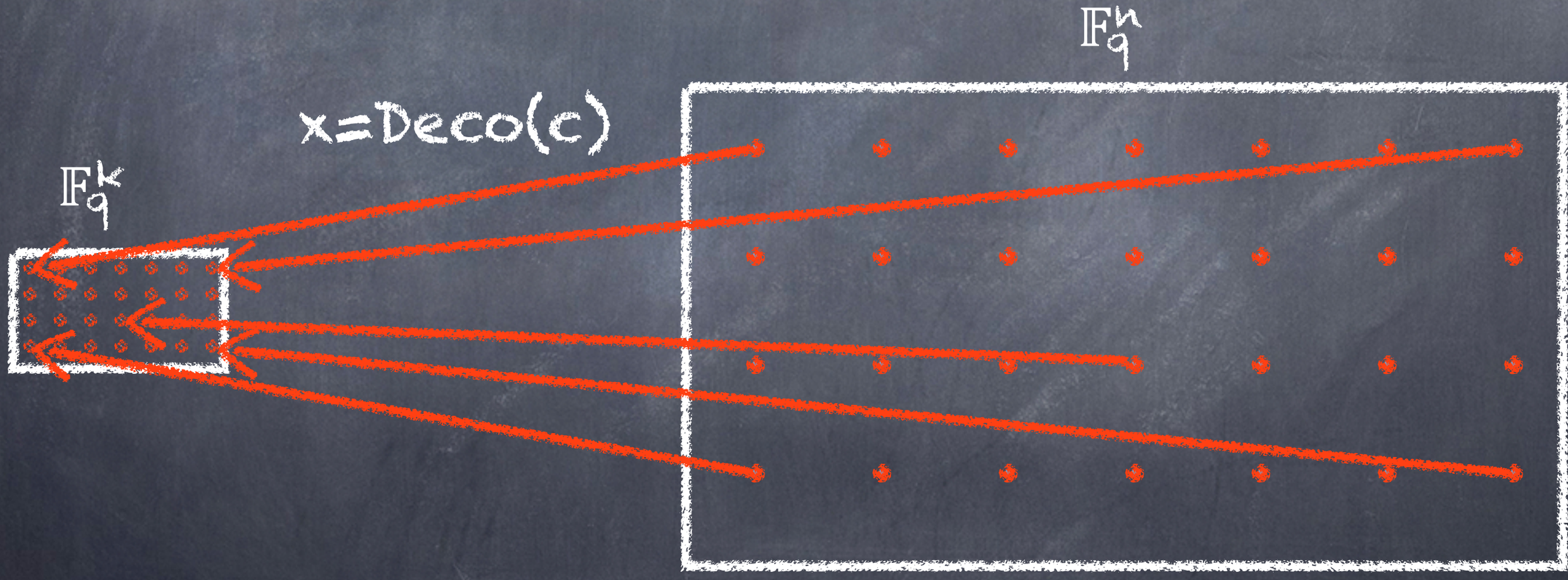
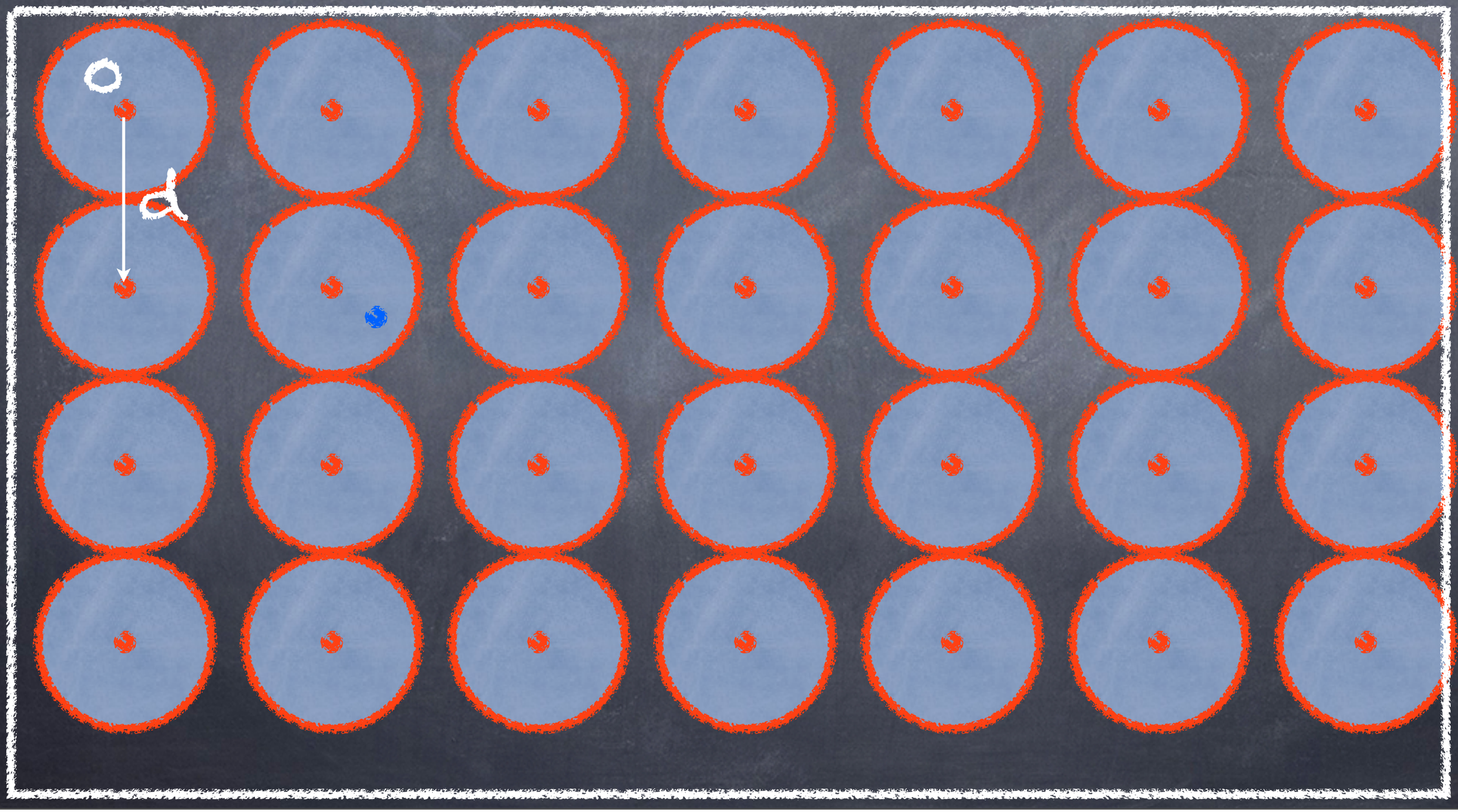


error detection vs error correction

$\mathbb{F}_q^n$



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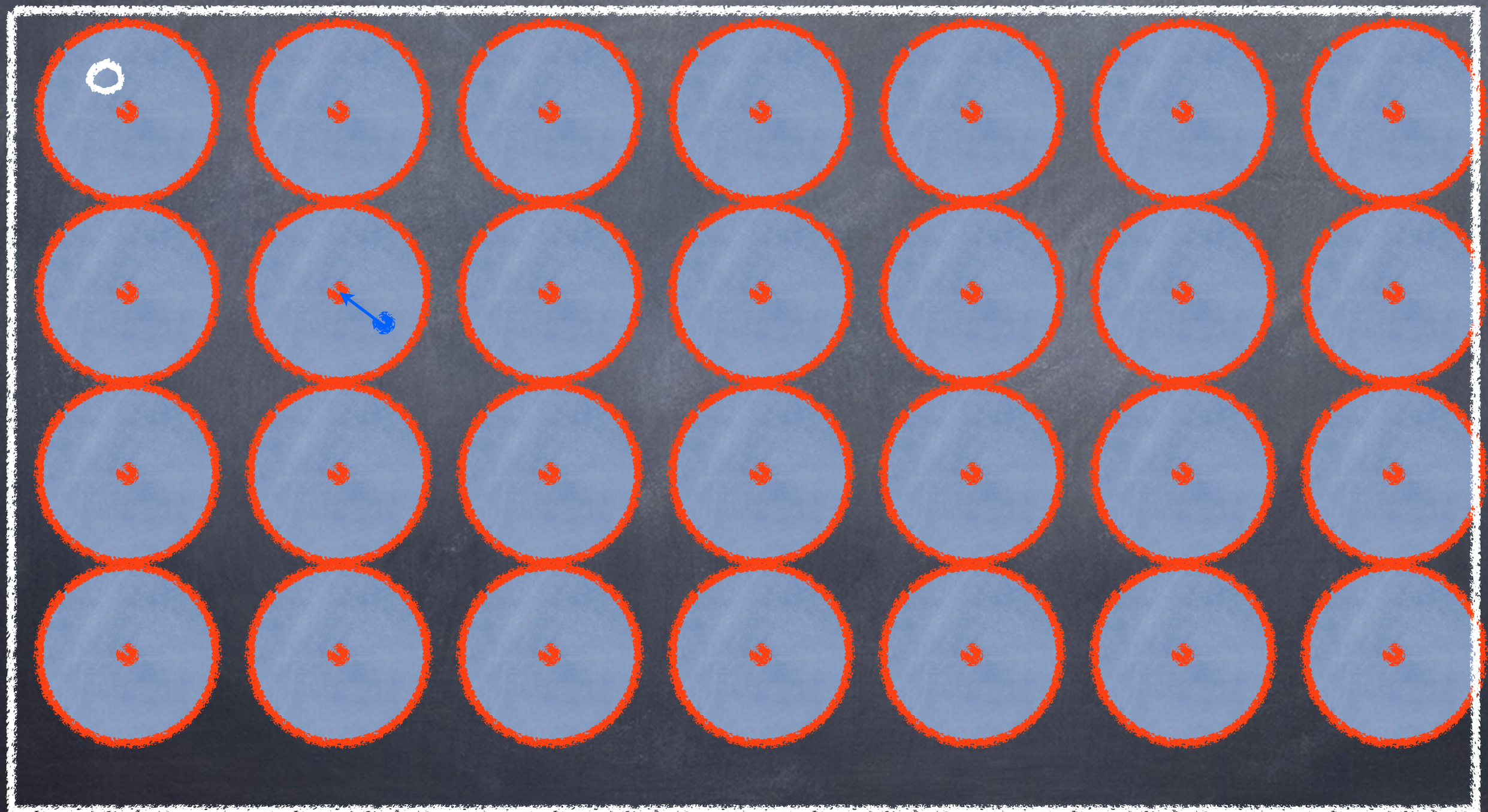

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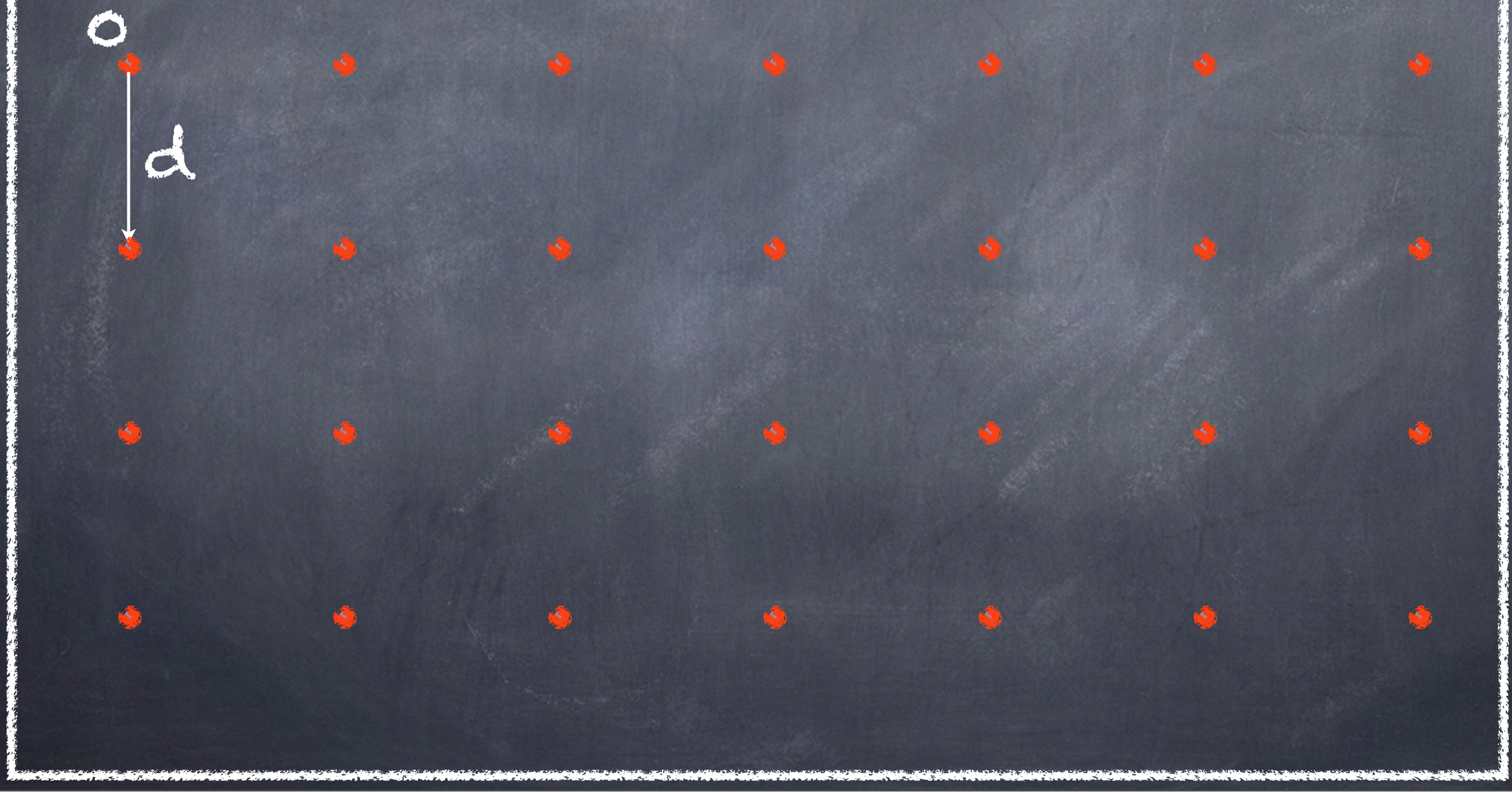

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basic Linear Coding theory

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- an kxn generating matrix G s.t. $C=\{ xG \mid x \in \mathbb{F}_q^k \}$
- an nx(n-k) parity check matrix H s.t. $C=\{ c \in \mathbb{F}_q^n \mid Hc=0 \}$.

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- Let C be an [n,k,d] linear code defined via an kxn generating matrix G or an nx(n-k) parity check matrix H and let w be $1 \leq w \leq n$

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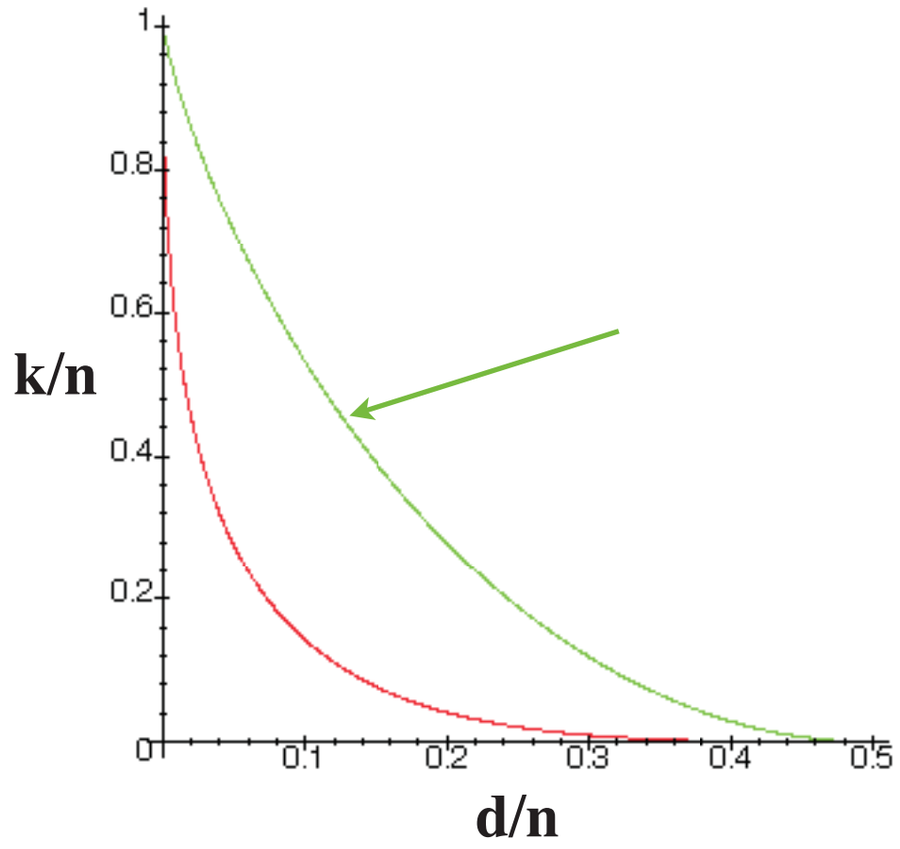
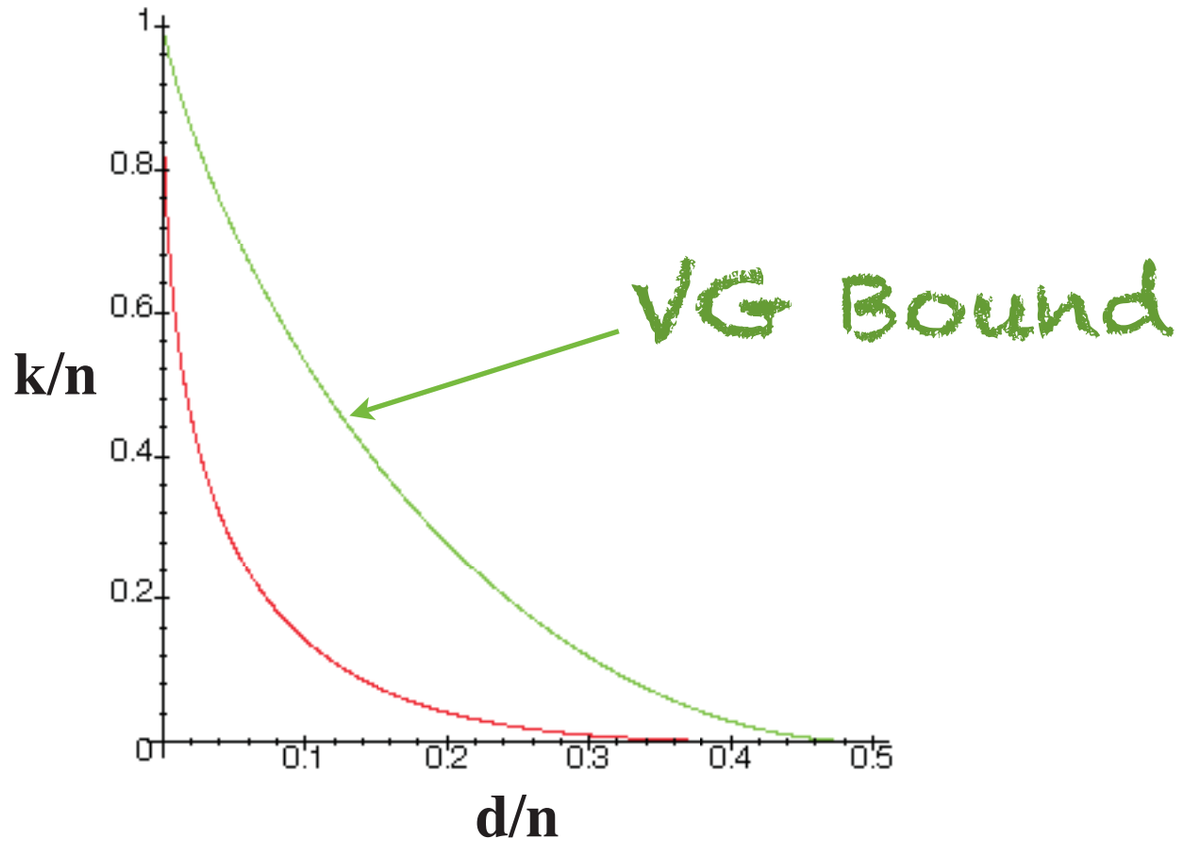


Varshamov-Gilbert Bound

basic Linear Coding theory



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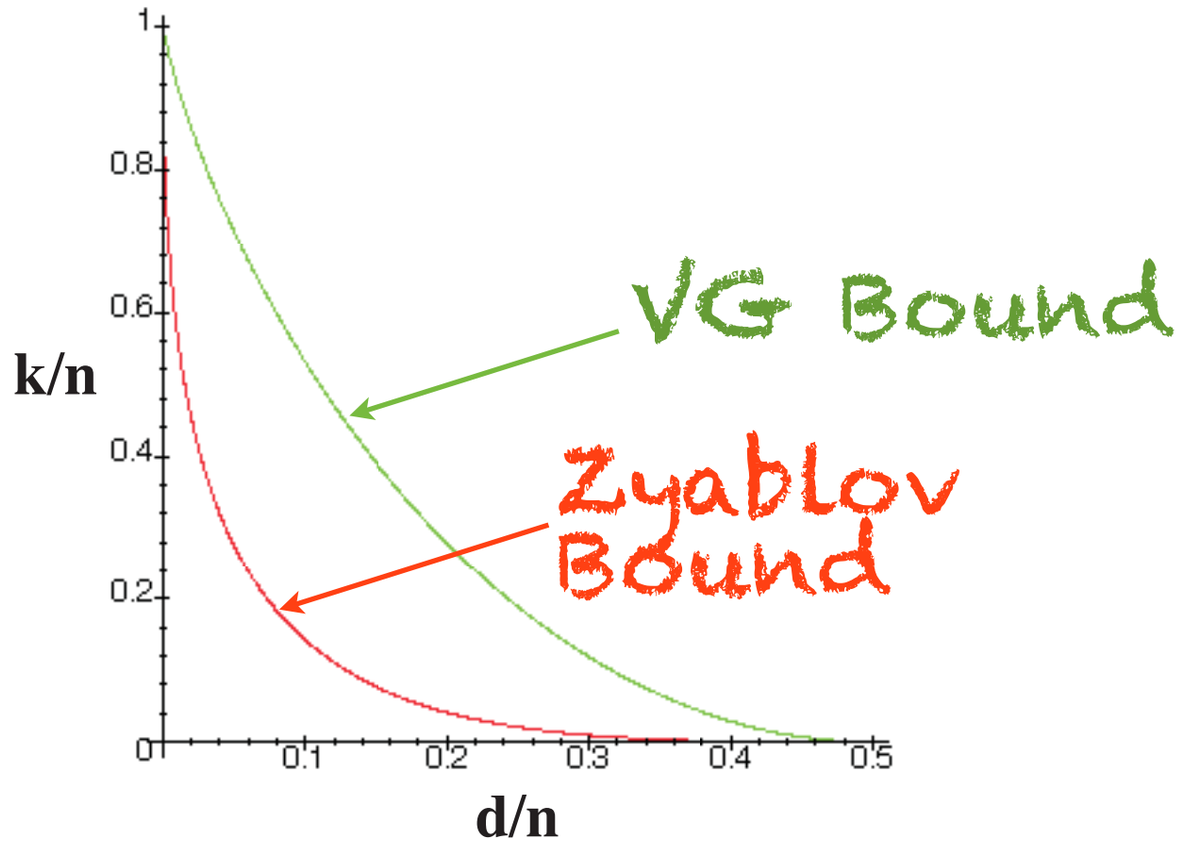

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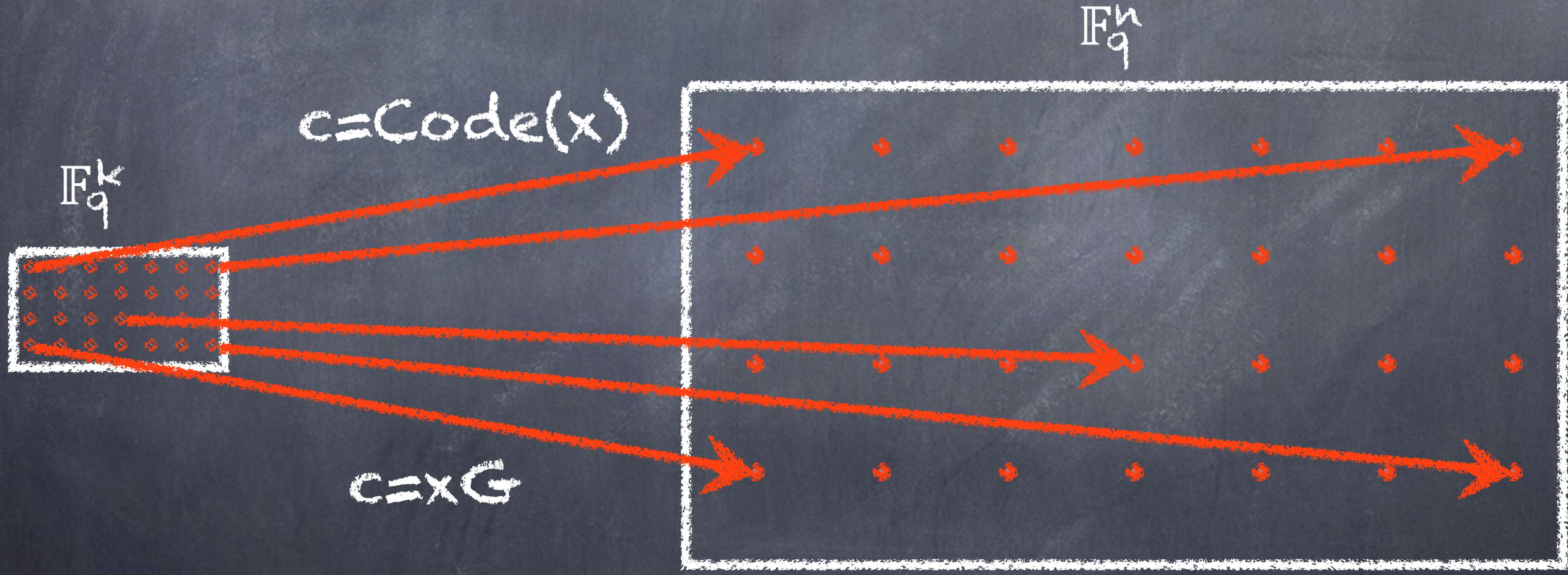
coding vs decoding



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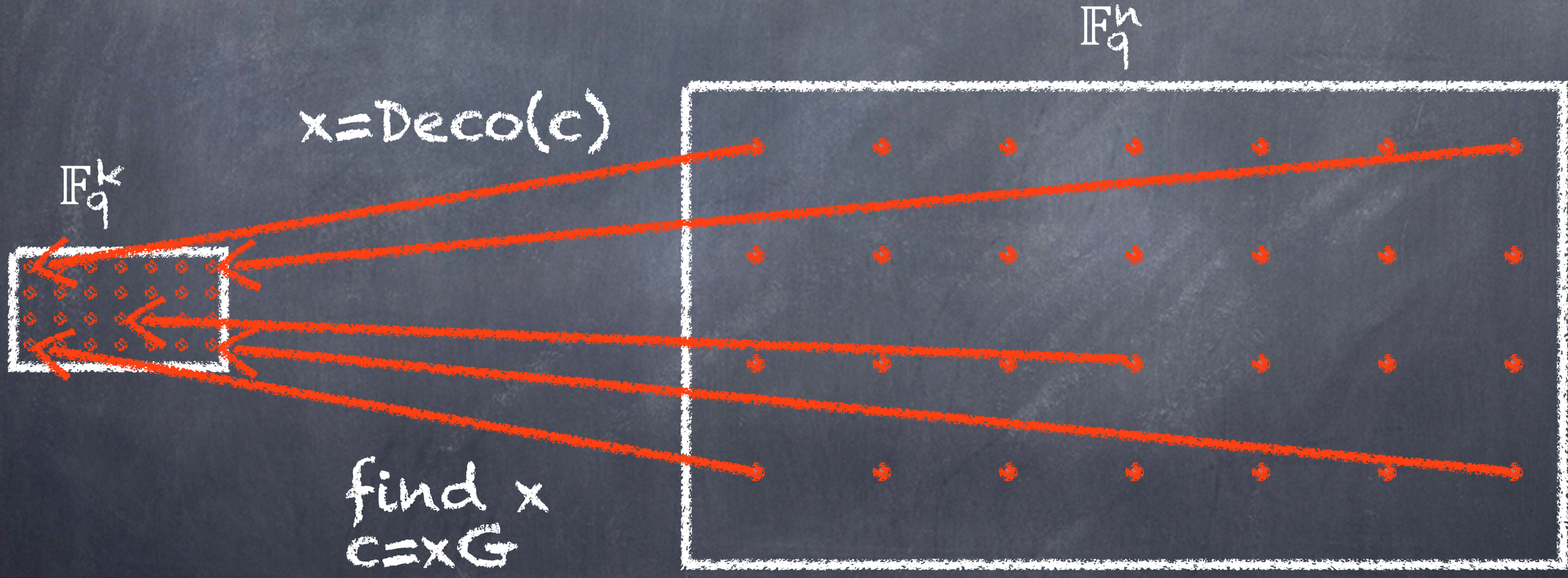
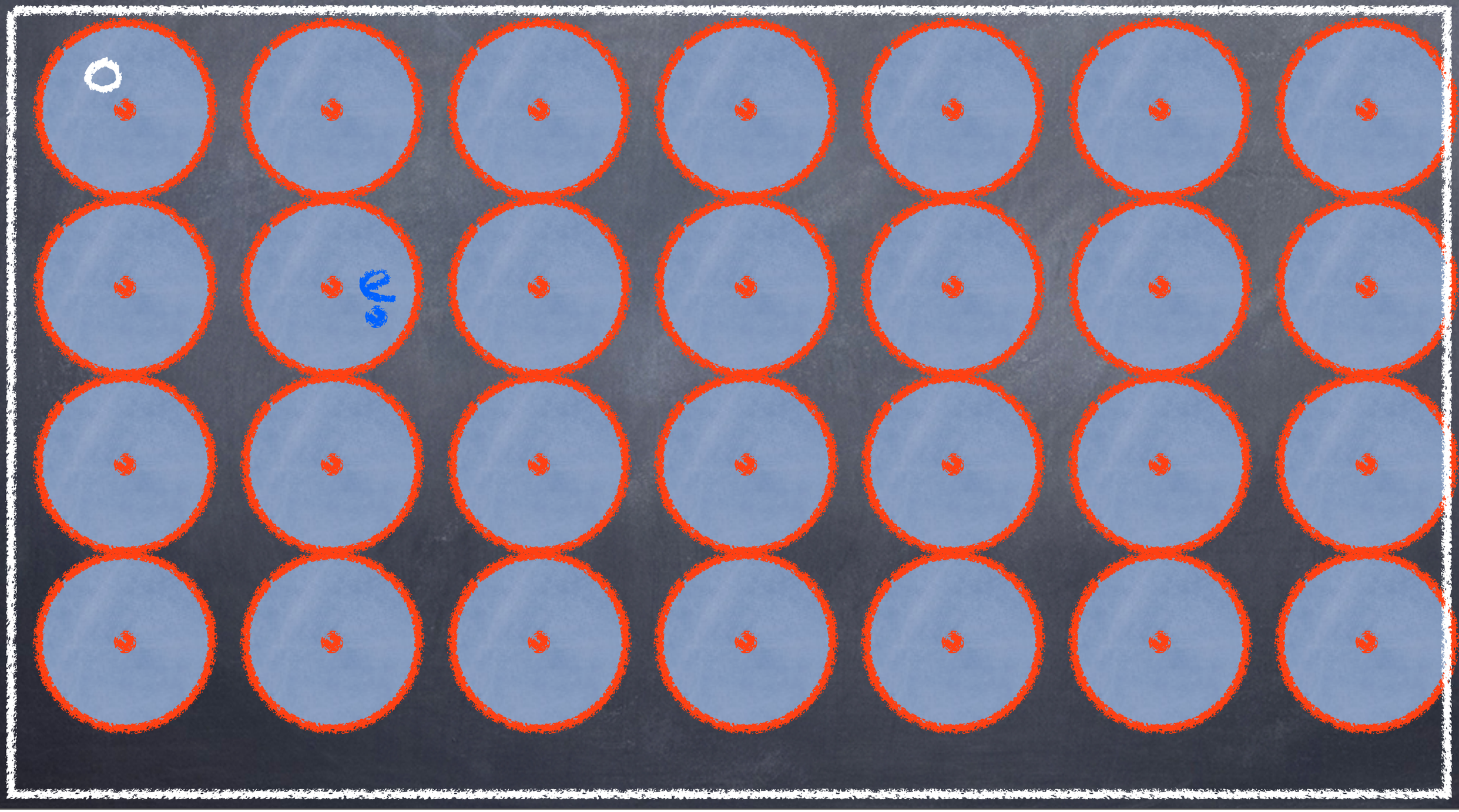


error detection vs error correction

$\mathbb{F}_q^n$



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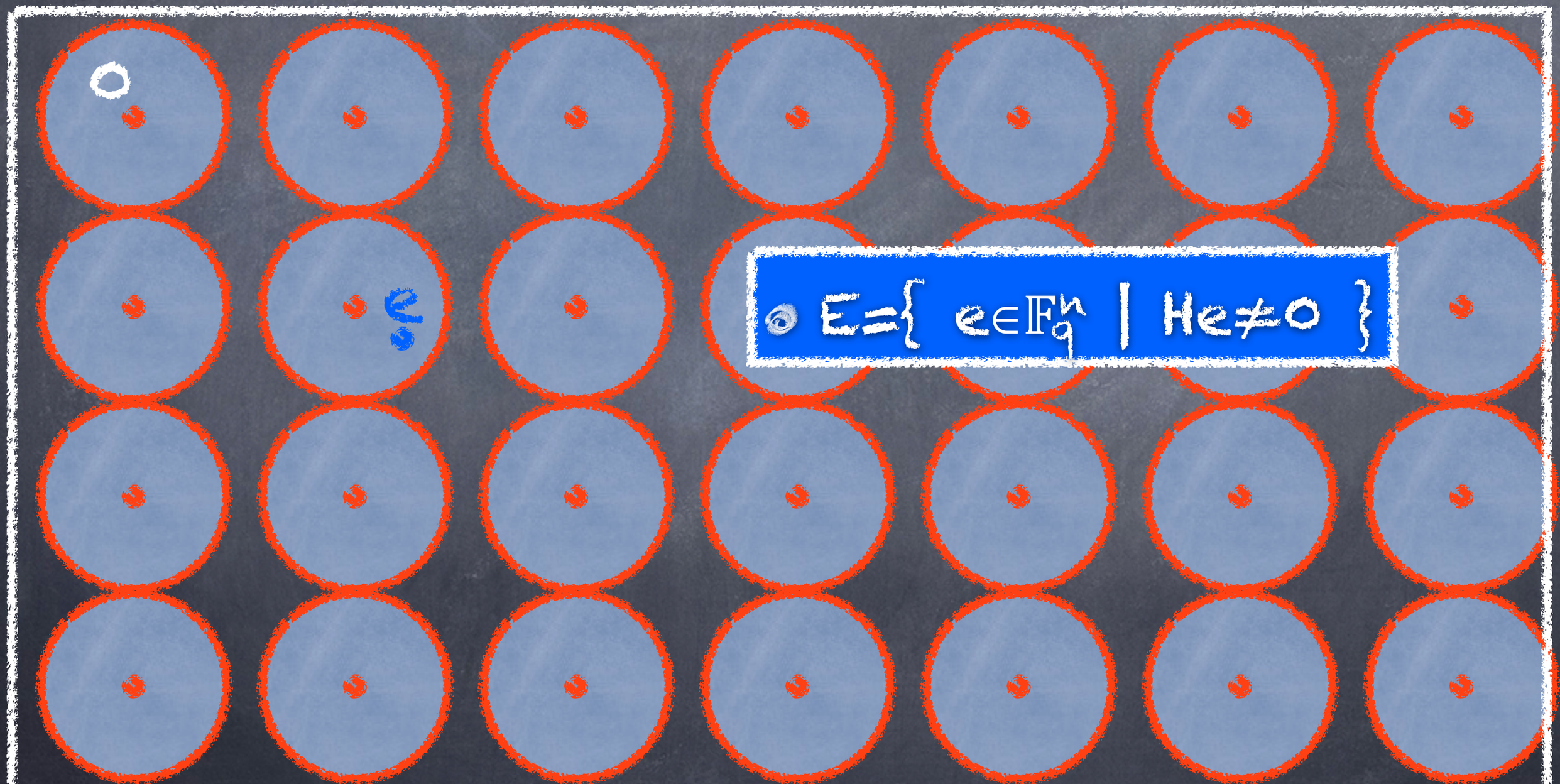

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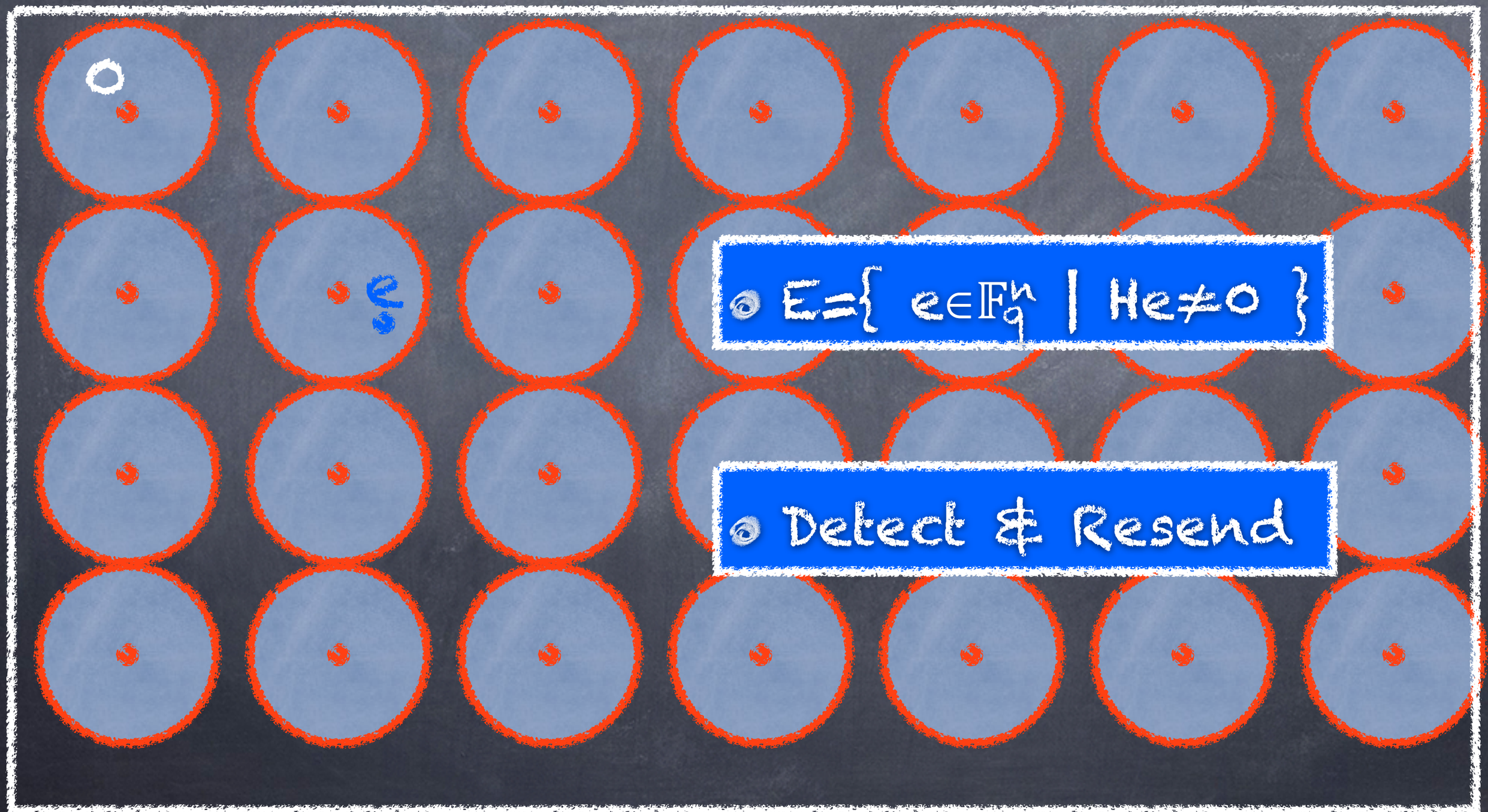

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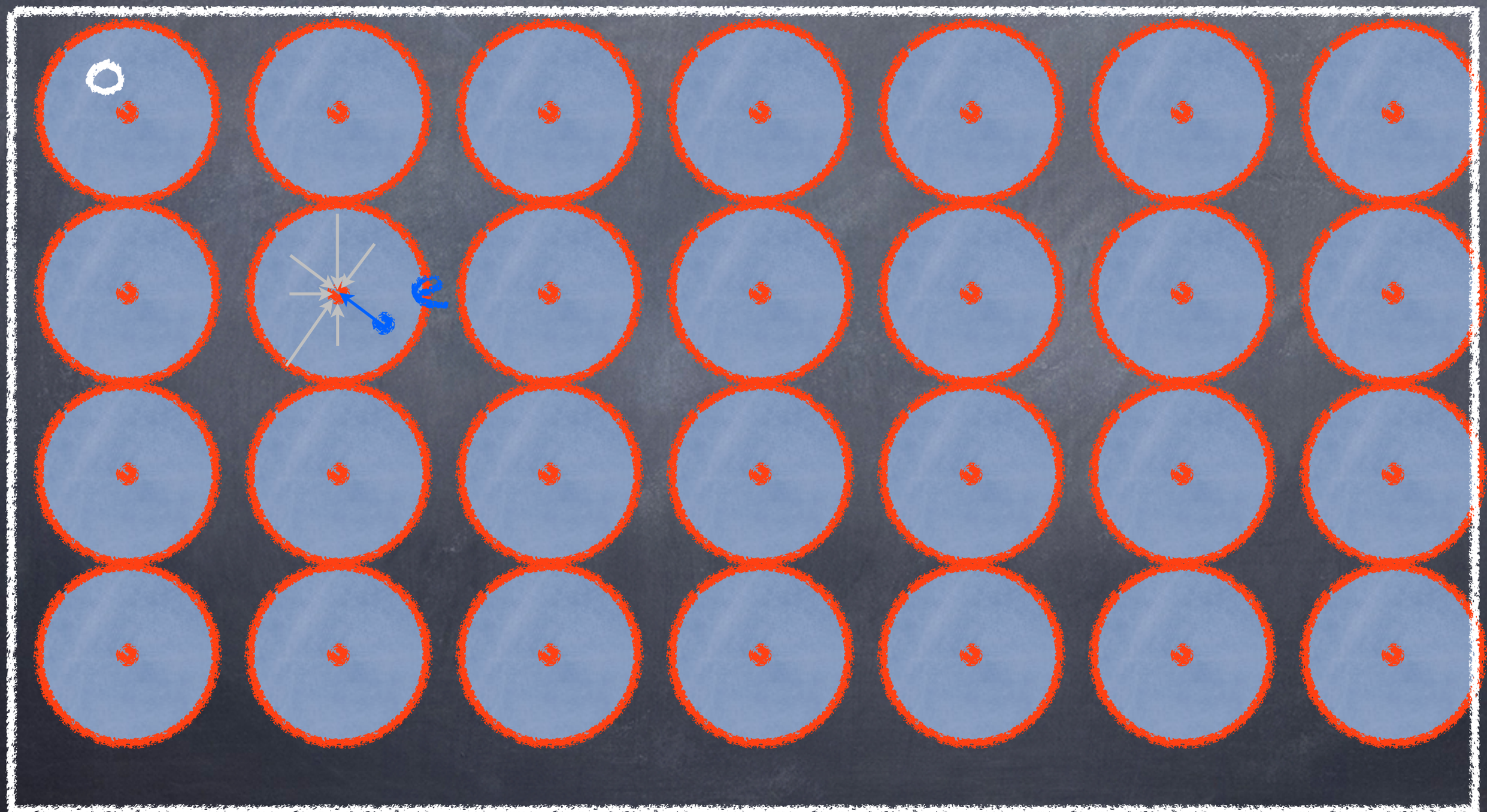

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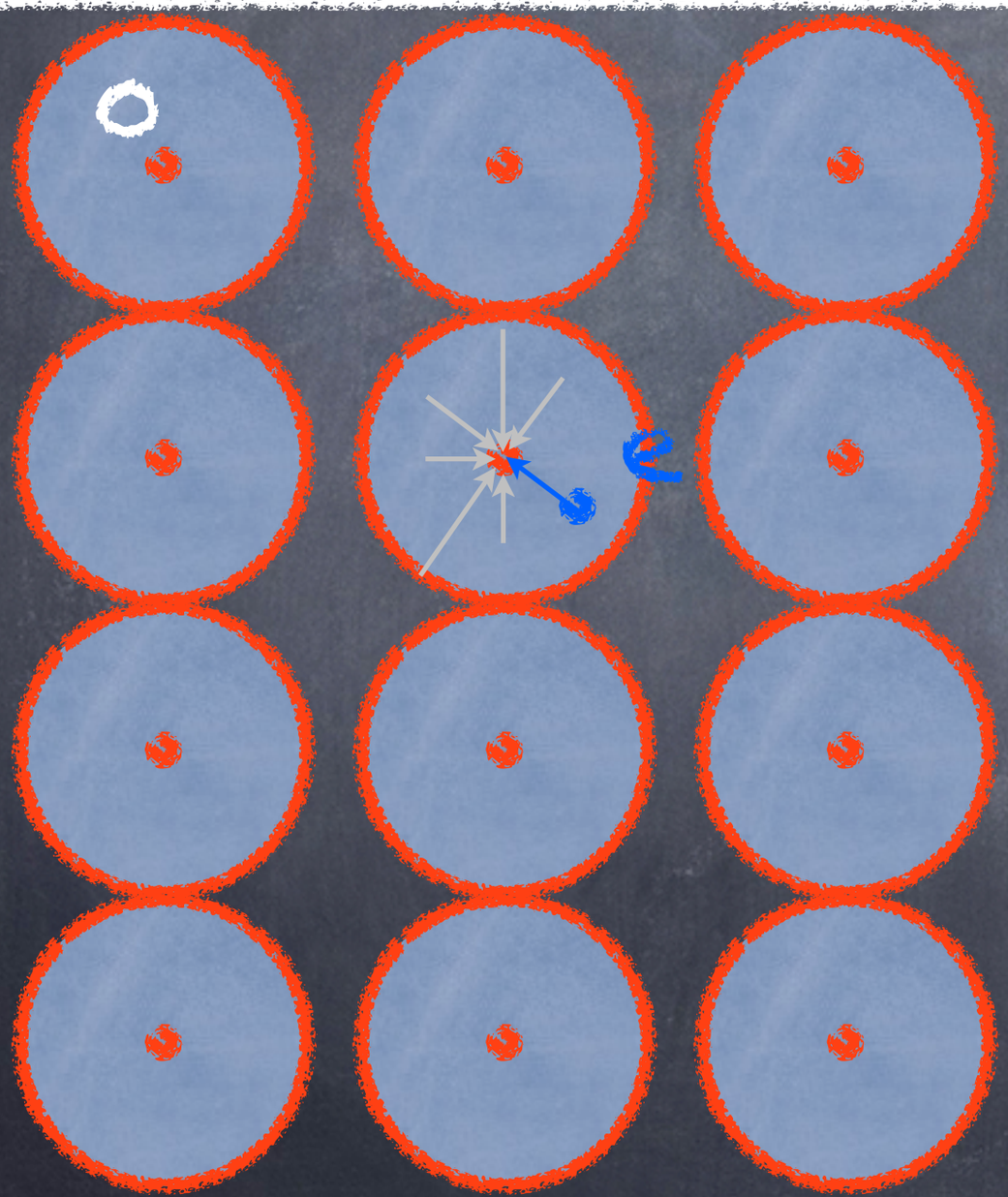
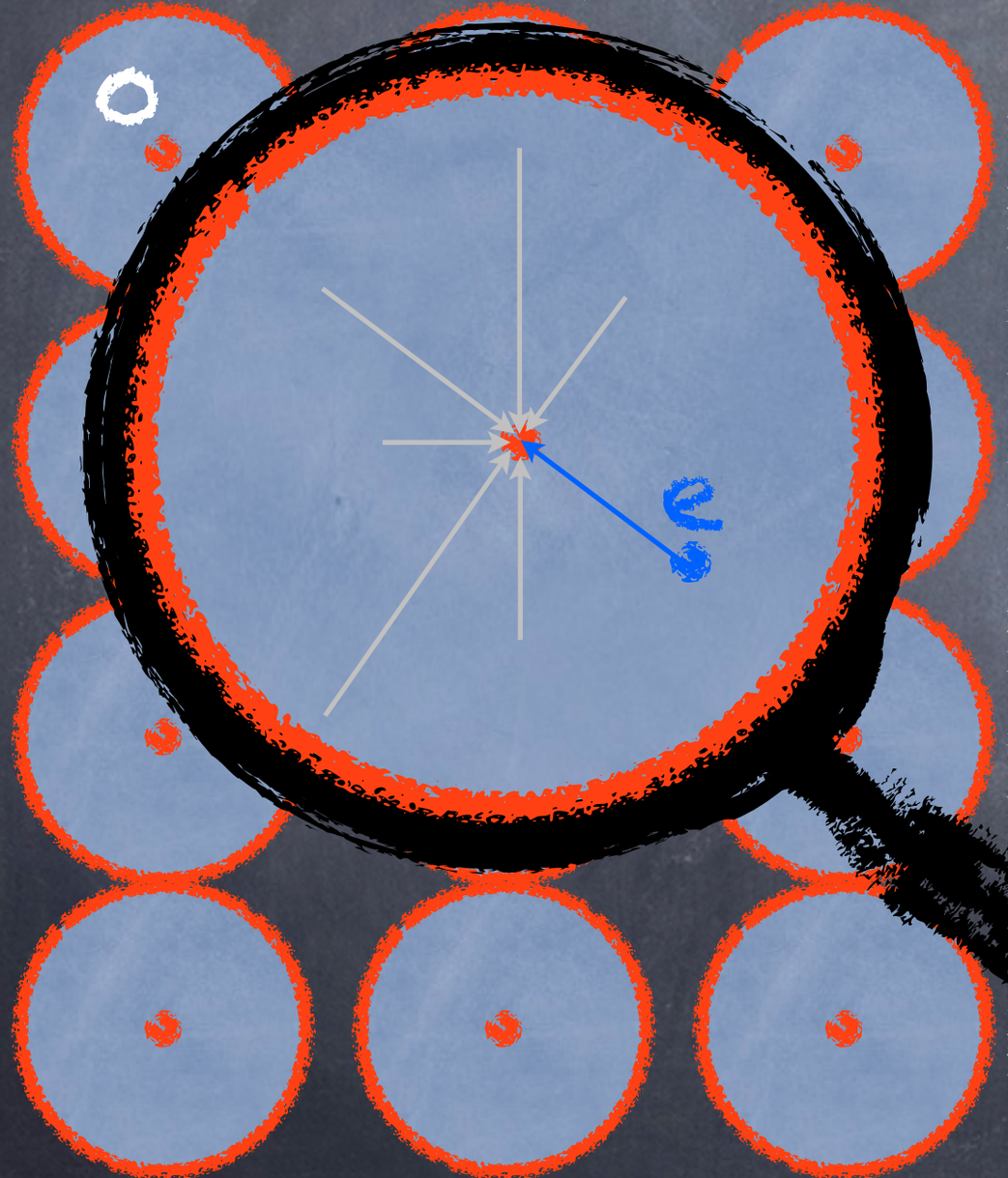


- Finding $c \in C$ s.t. $d_H(c,e)$ is minimum
- is NP-hard
- is highly structured and non trivial in many special cases.

error detection vs error correction

$\mathbb{F}_q^n$



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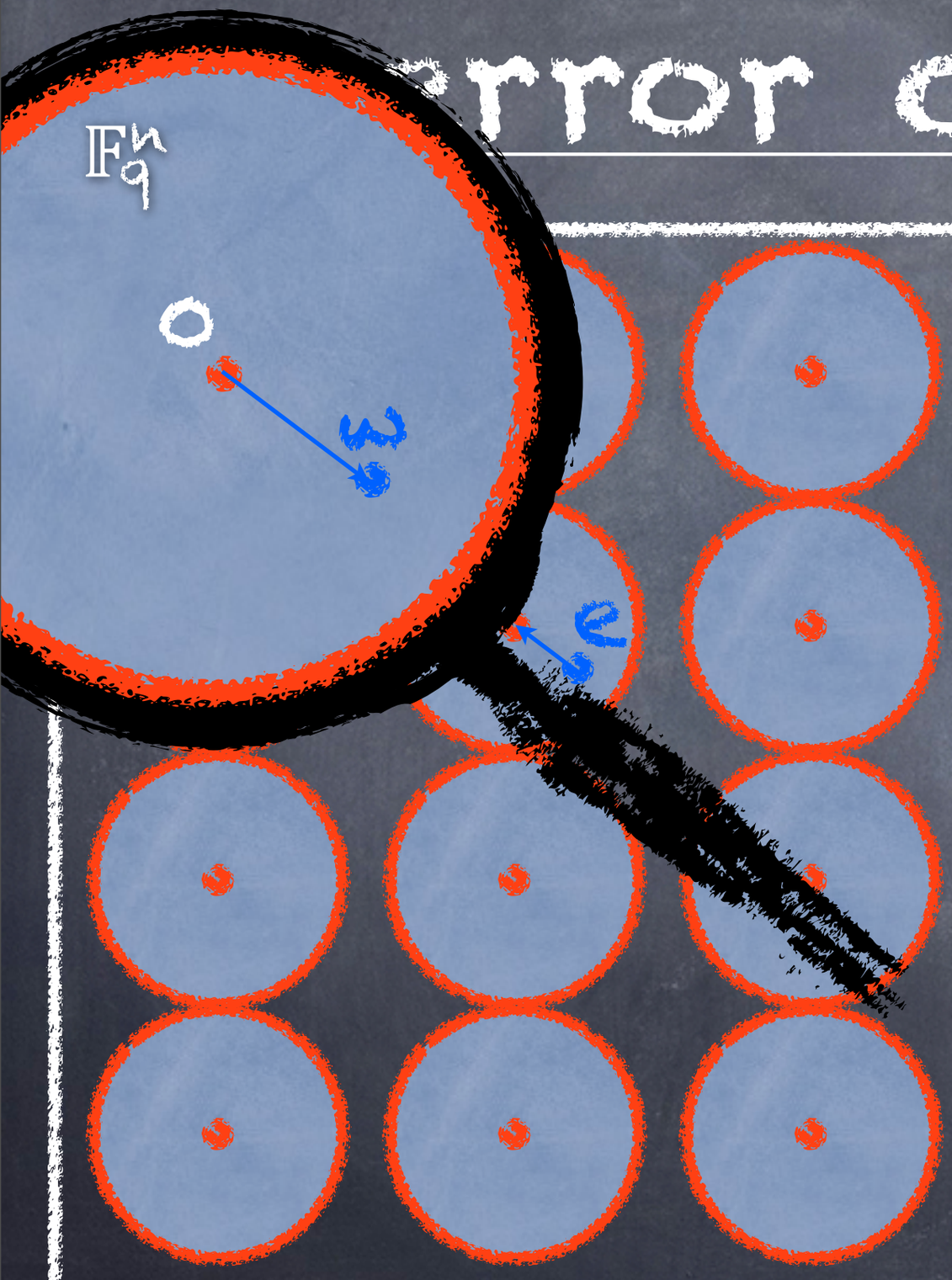
error detection vs error correction



- Finding w s.t. wt(w) is minimum & Hw=He
- is NP-hard
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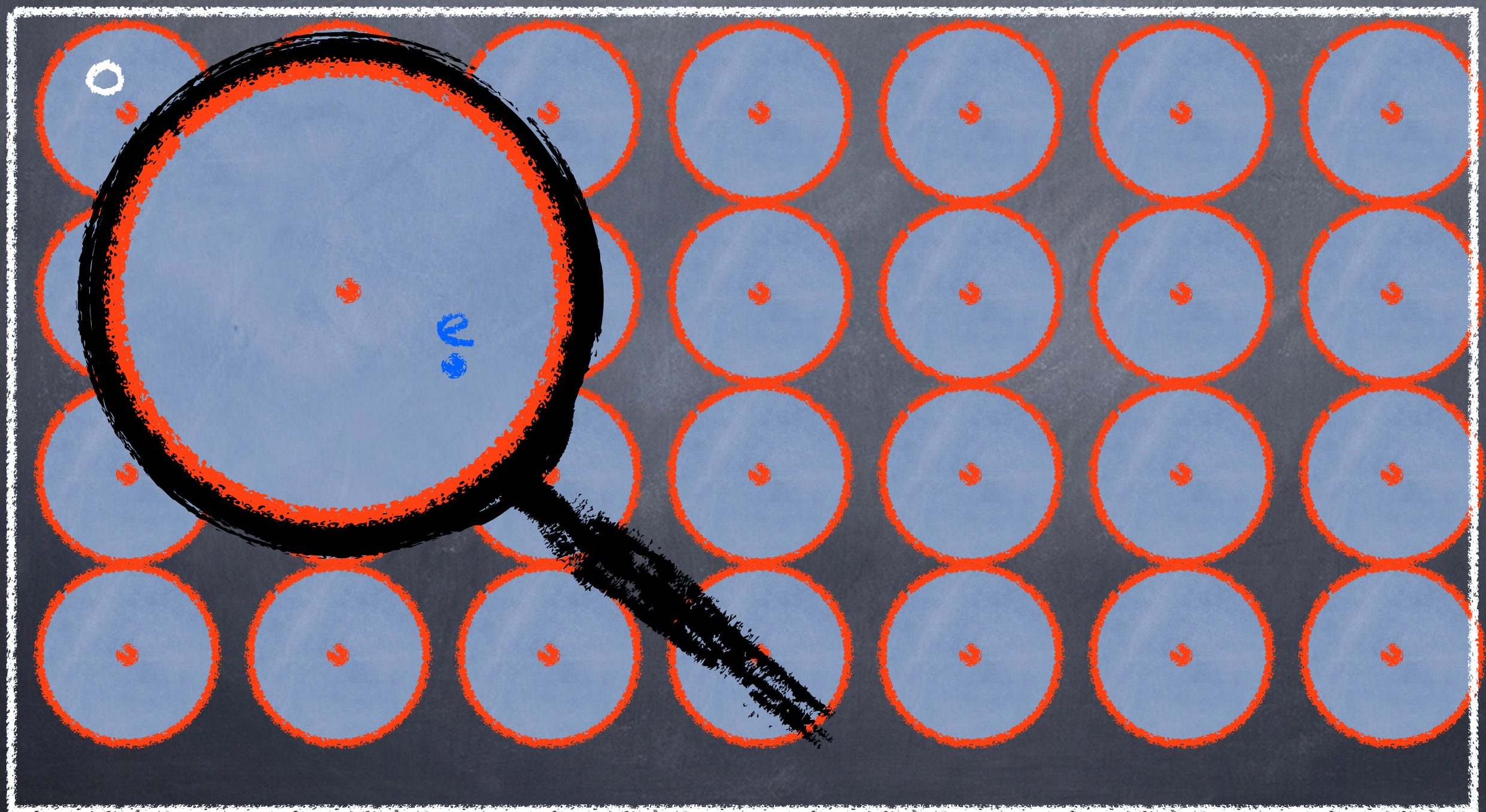
error detection vs error correction

- (inefficient correction procedure)
- To correct a word w upto $d-1/2$ errors applied to an [n,k,d] codeword
- try all error patterns e upto $d-1/2$ errors
- check whether w-e is in the code.

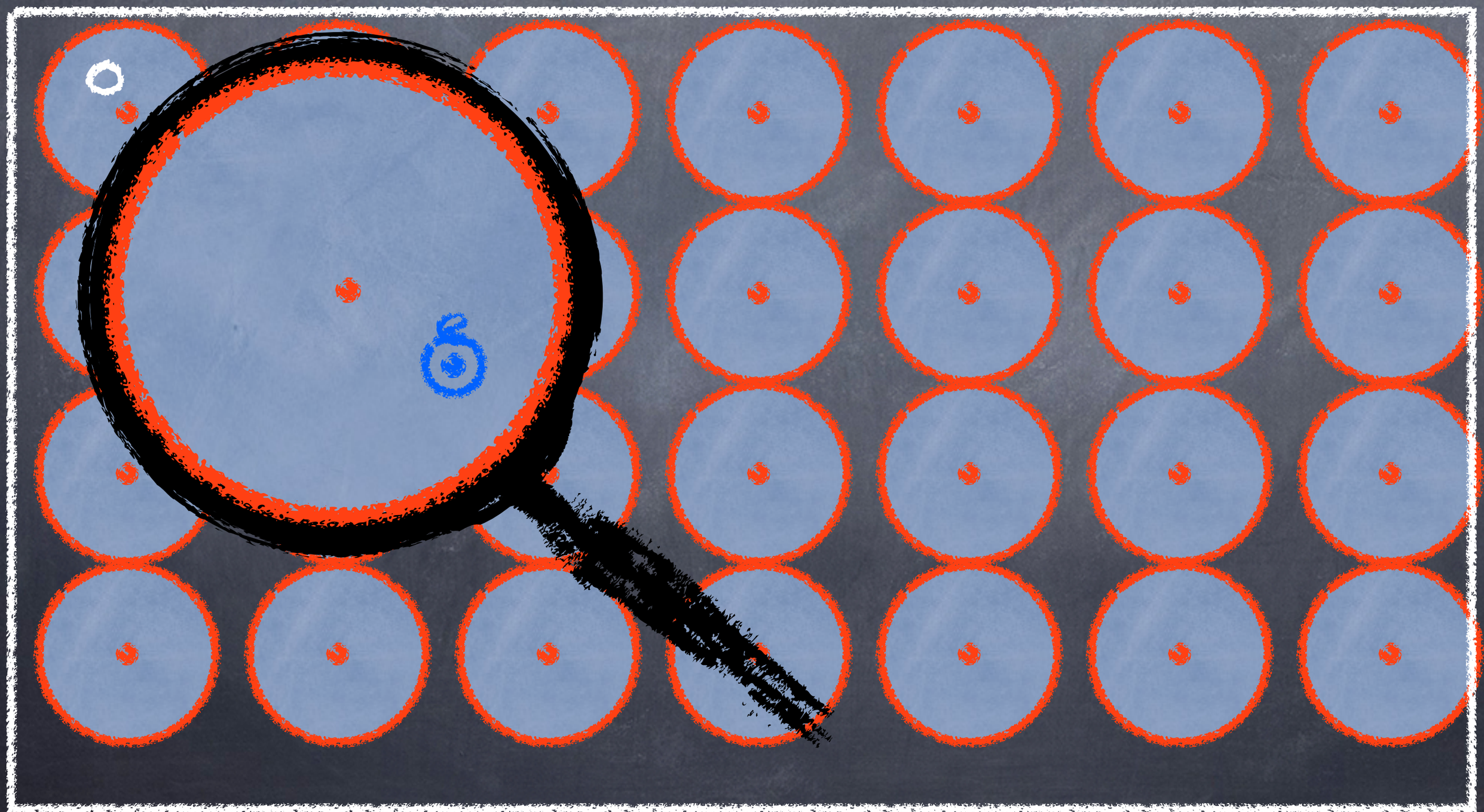
error detection vs error correction

$\mathbb{F}_q^n$



error detection vs error correction

$\mathbb{F}_q^n$



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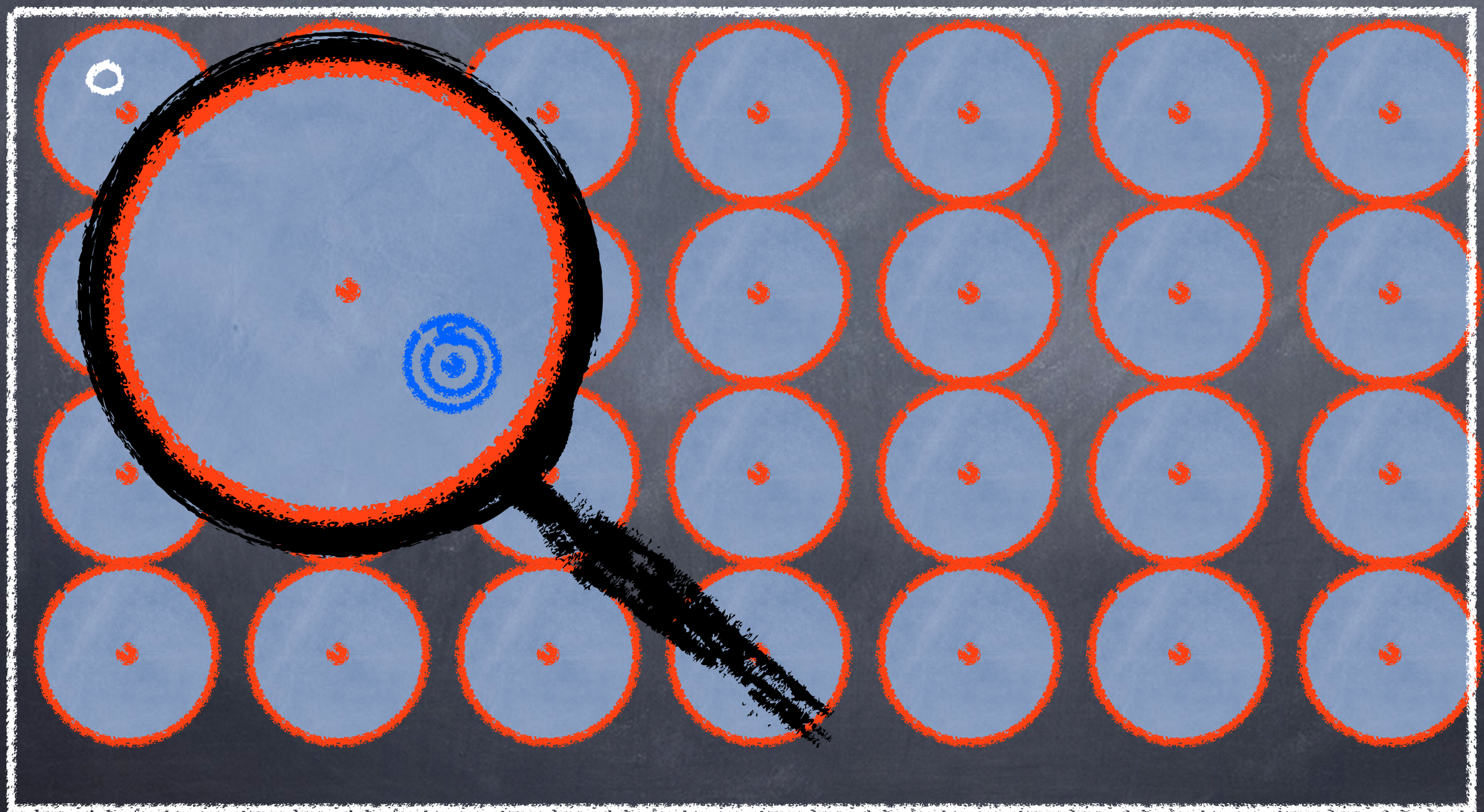

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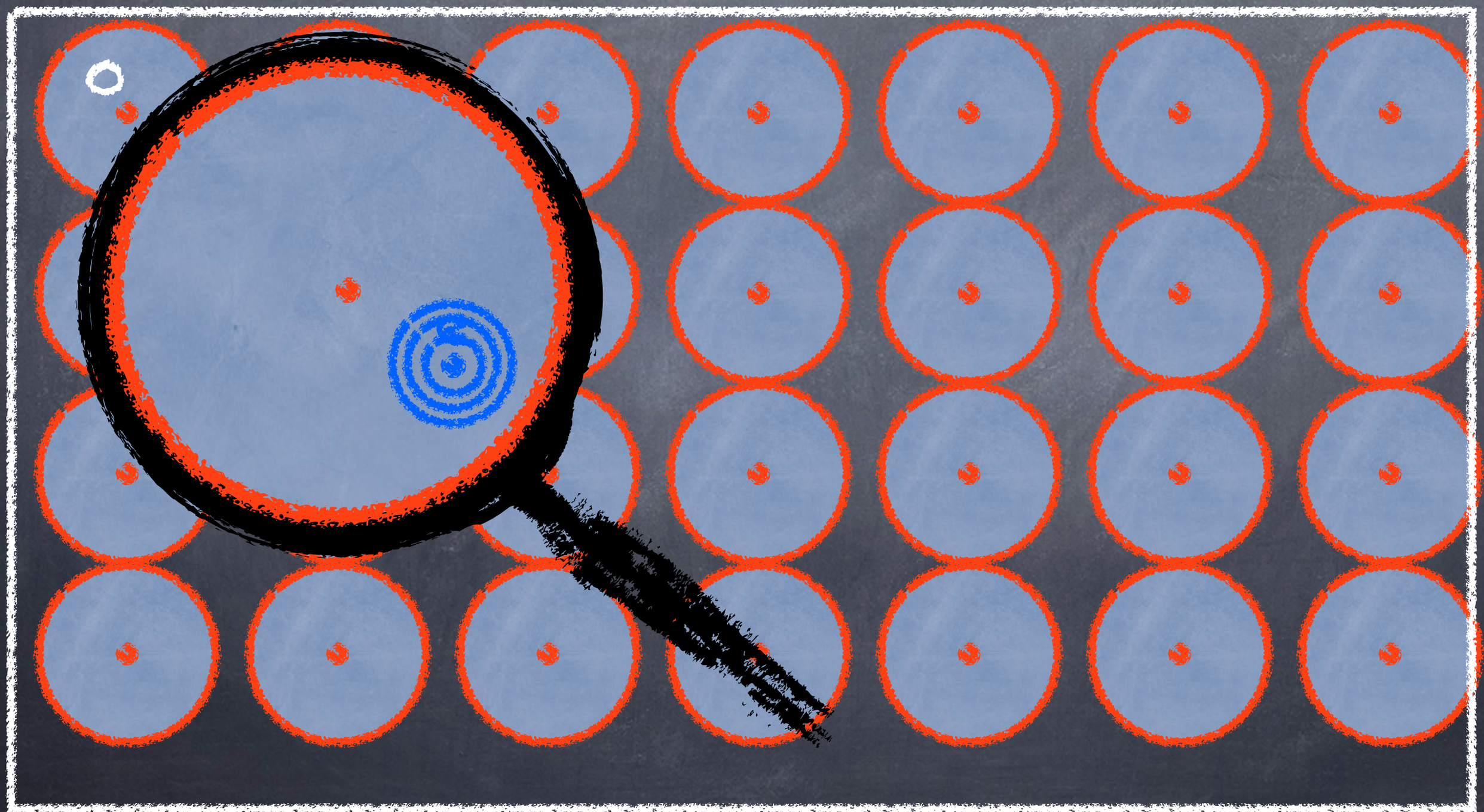

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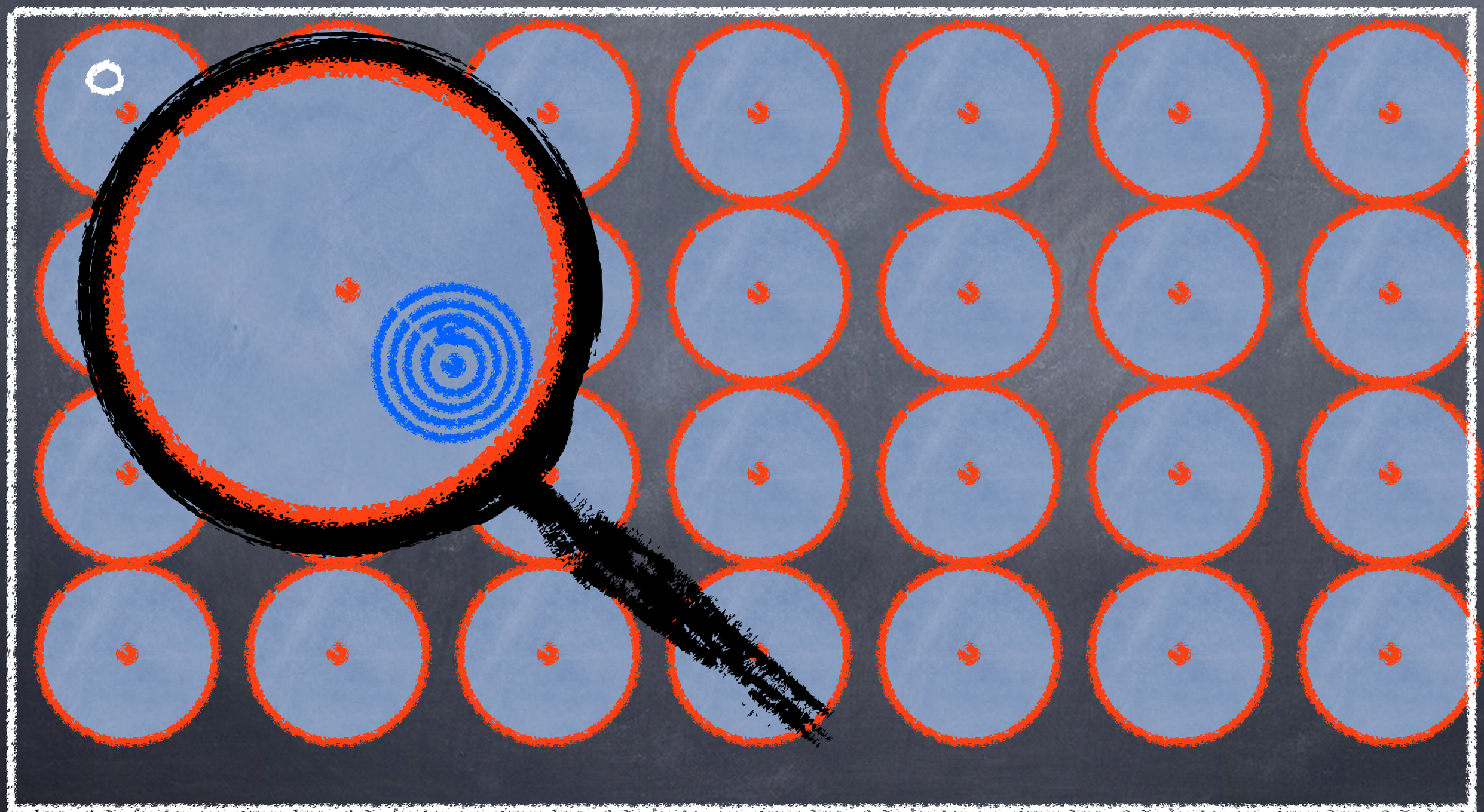

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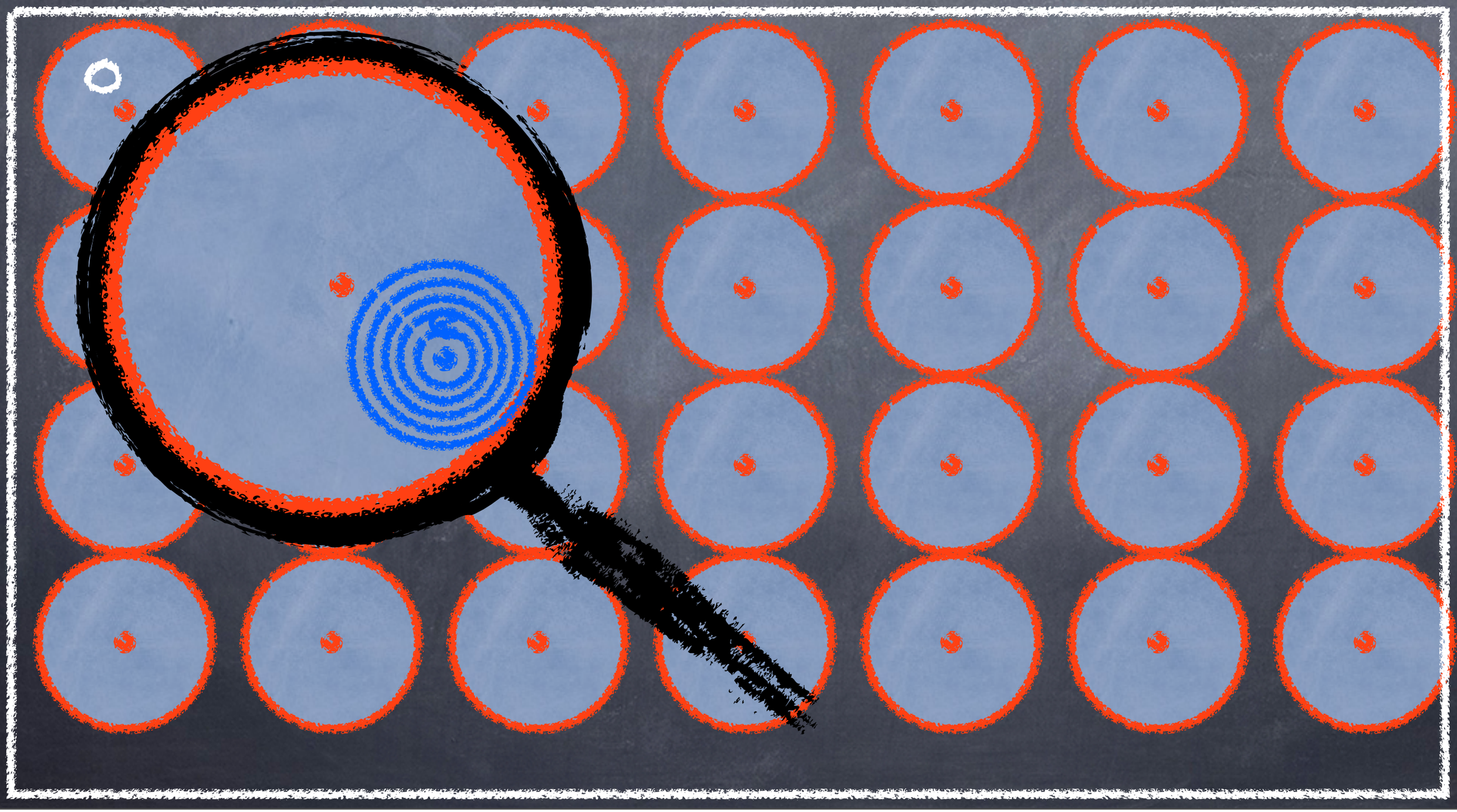

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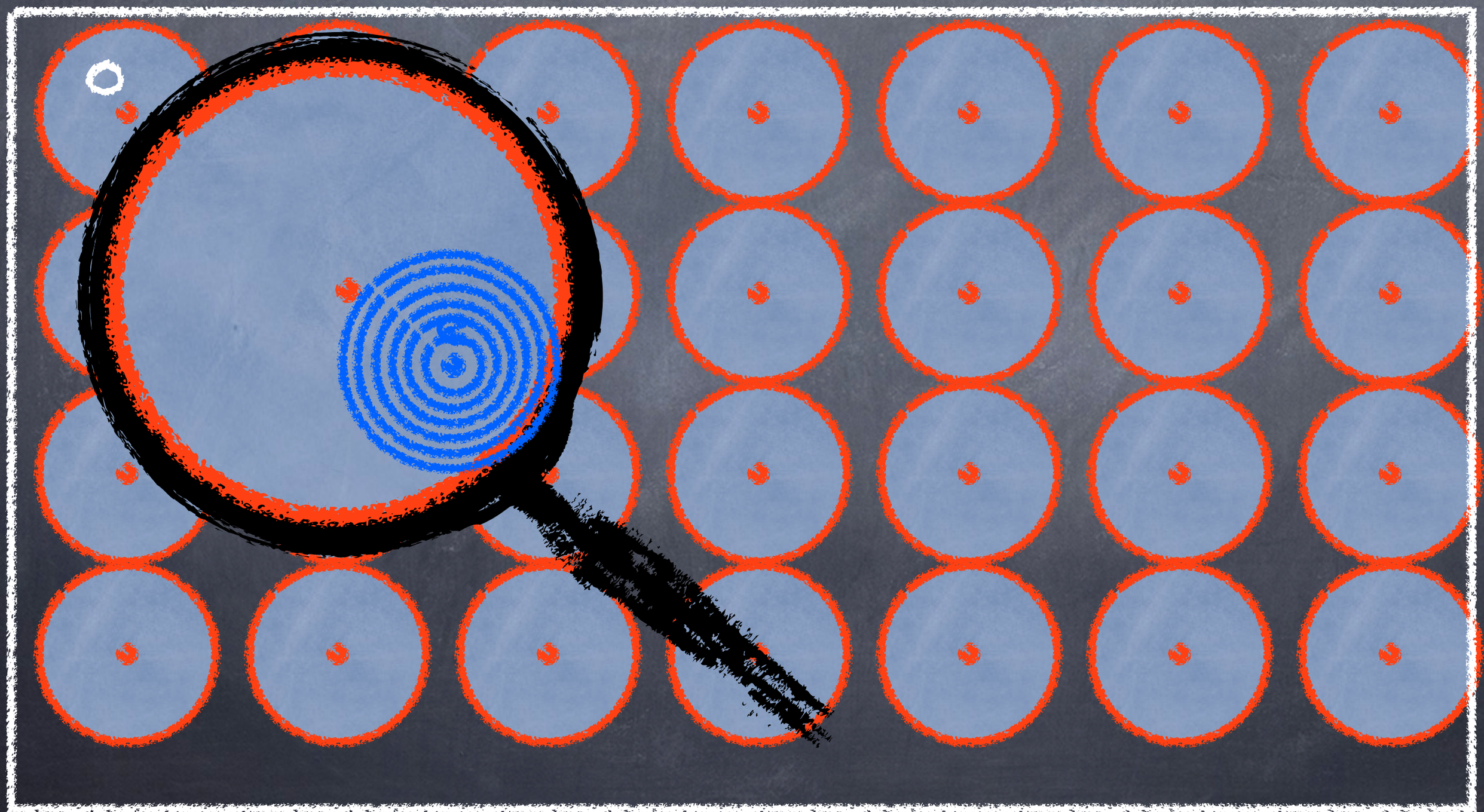

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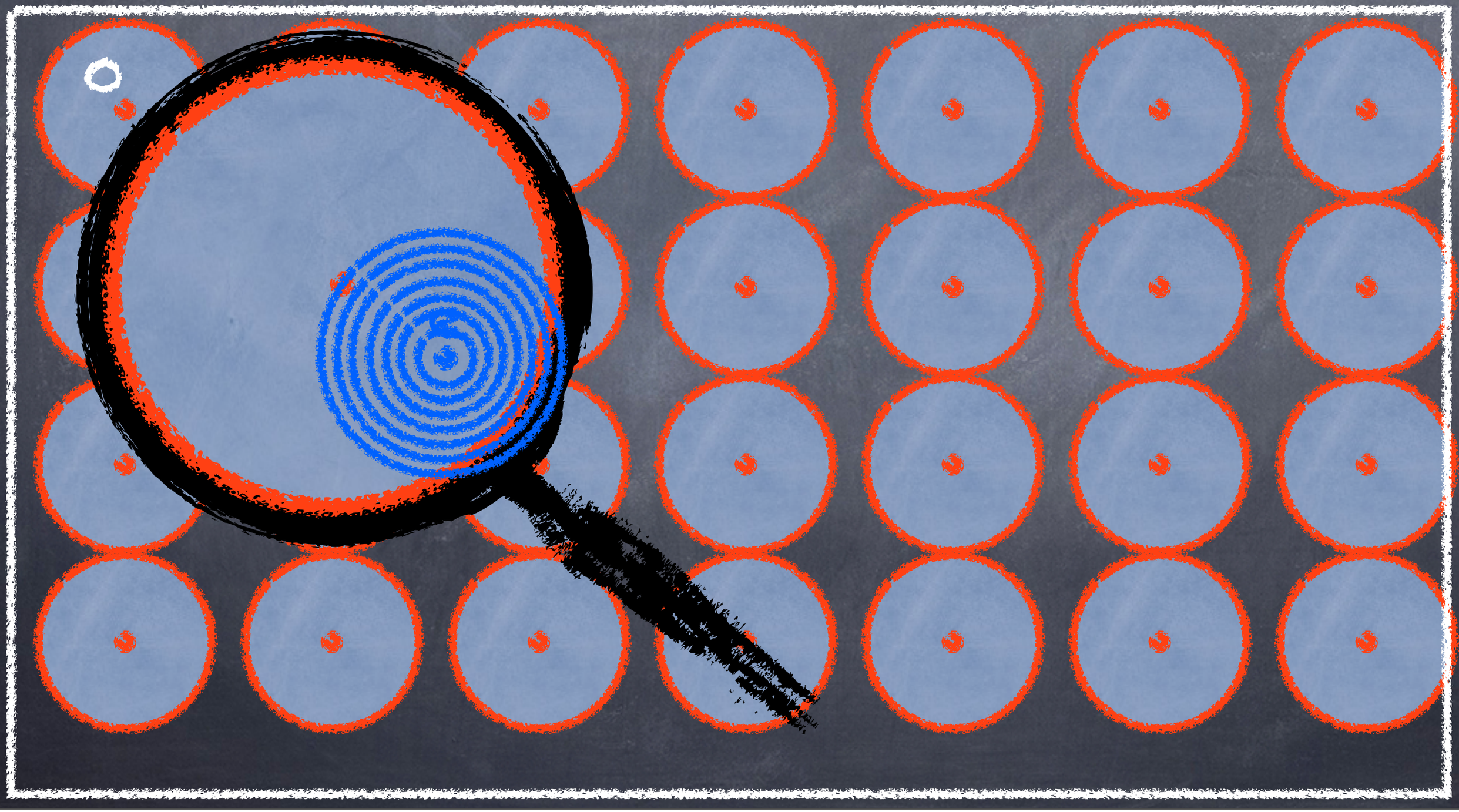

error detection vs error correction

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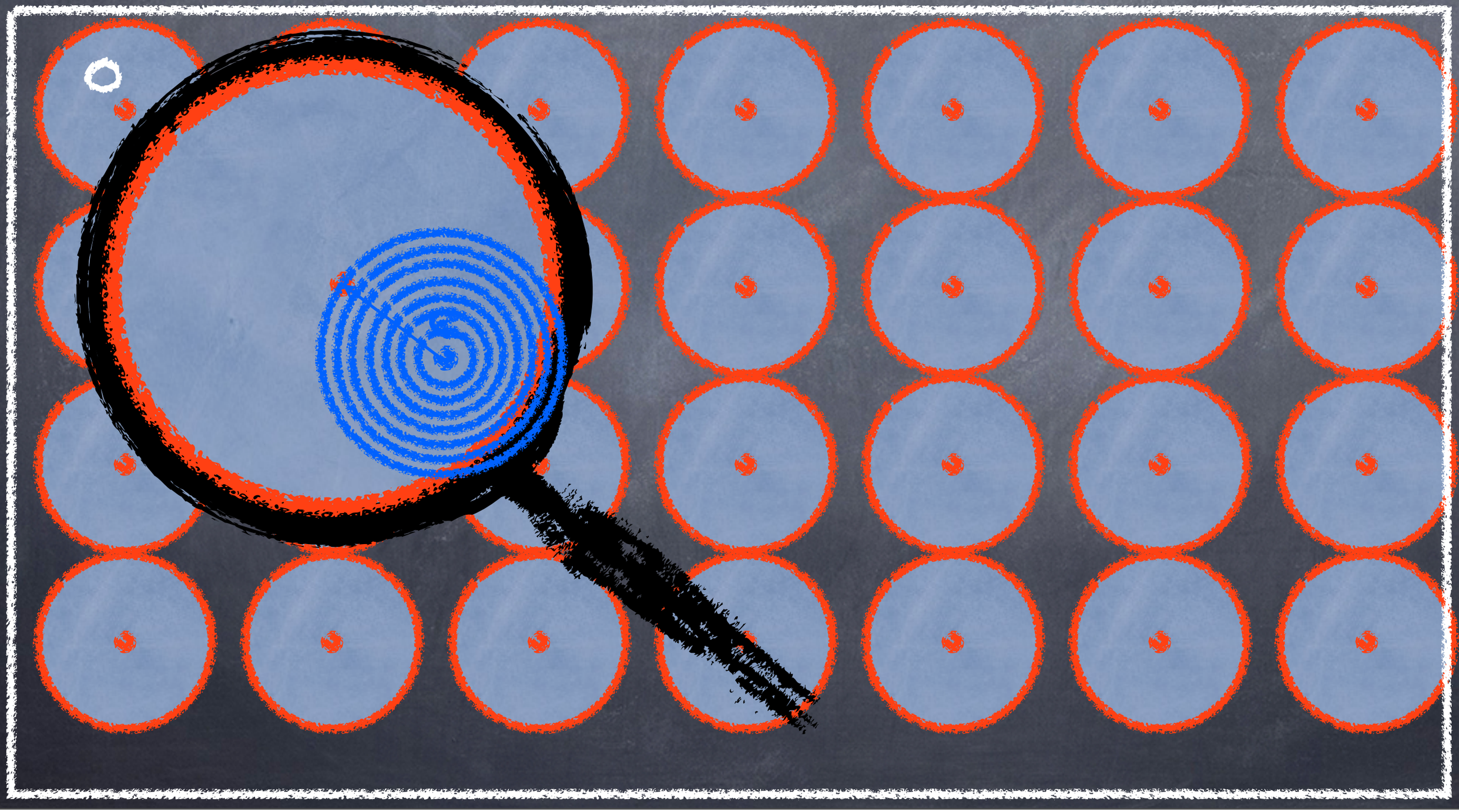

$[2^r-1, 2^r-r-1, 3]$ Hamming codes

- Hamming code: [7,4,3]

$$H = \begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \end{pmatrix} \quad \begin{matrix} \\ \\ \end{matrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$C = \{\, c \mid Hc = 0 \,\}$$

[7,4,3]
Hamming code

- $$H = \begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \end{pmatrix}$$

$H(c+e_i) = 0+He_i = He_i = i$ such that

$e_i = (00...1...0)$

123...i...7

Correcting [7,4,3] Hamming code

- $$H = \begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \end{pmatrix}$$

$H(w) = i$ then $c = w - e_i$ is the nearest codeword...

Reed-Solomon Codes

Reed-Solomon Codes

- an [n,k,d] Reed-Solomon code C over a field $\mathbb{F}_n$ is the set of n^k codewords of the form

($p(0), p(\alpha^1), \ldots, p(\alpha^{n-1})$) where p is any polynomial of degree less than k and $0, \alpha^1, \ldots, \alpha^{n-1}$ is a list of the elements of $\mathbb{F}_n$.

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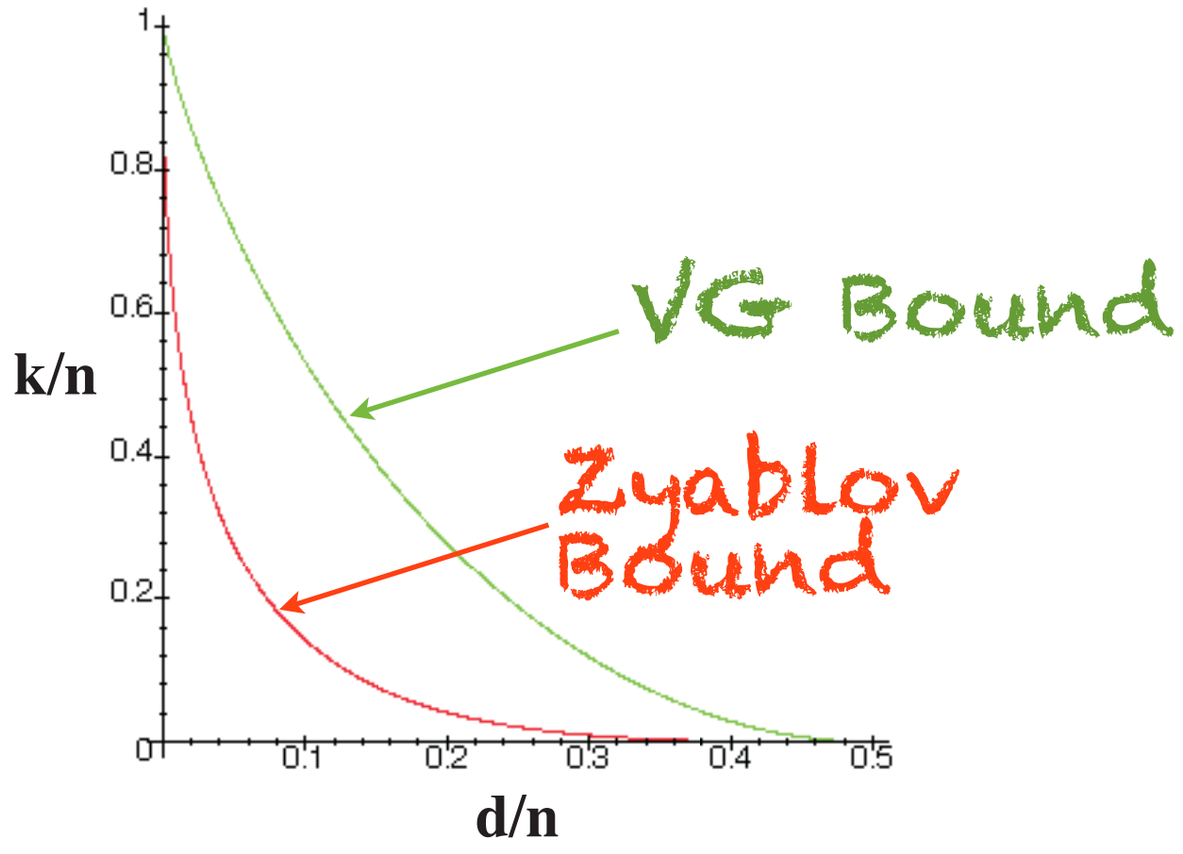
- Yeilds Zyablov Bound

Reed-Solomon Codes

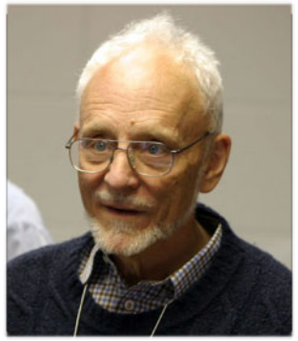
- ar… c
  o…
  co…
  (… s
  ar… than
  k… …e
  el…



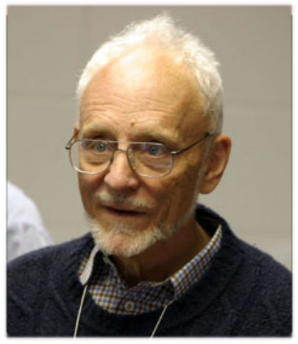
- Yeilds Zyablov Bound

Reed-Solomon Codes

- all [n,k,d] Reed-Solomon codes have d=n-k+1 since any codeword with at least k values = zero must be the all-zero codeword. (unique interpolation theorem)

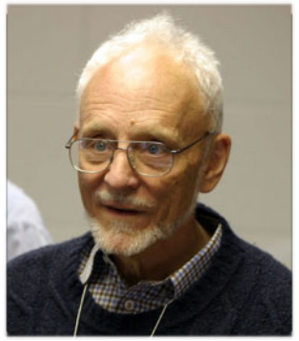


Berlekamp-Welch error correction



Berlekamp-Welch error correction

- (efficient correction procedure)



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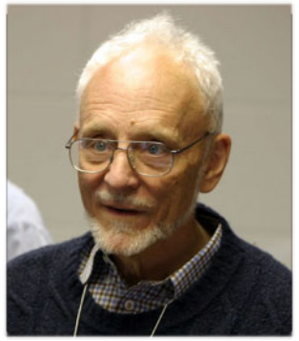
- To correct a word w upto (n-k)/2 errors applied to an [n,k,n-k+1] Reed-Solomon codeword,



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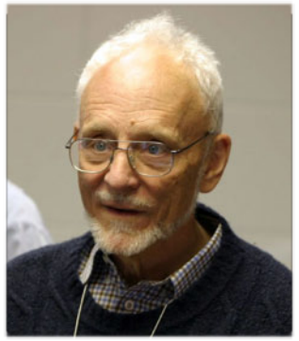

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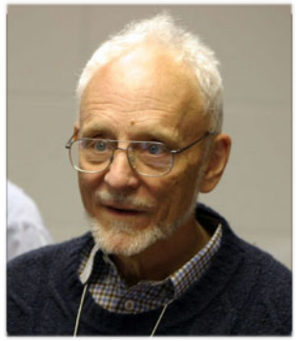

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- (efficient correction procedure)
- To correct a word w upto (n-k)/2 errors applied to an [n,k,n-k+1] Reed-Solomon codeword,
- find the positions in error,
- interpolate p using correct positions
- substitute each error by the correct value of p(x).

Generalized RS

- an [n,k,d] Generalized RS code C over a field $\mathbb{F}_n$ is a set of n^k codewords of the form $(z_1p(a_0), z_2p(a_1), \ldots, z_np(a_{n-1}))$, $z_i \neq 0$ where p is any polynomial of degree less than k and $a_0, a_1, \ldots, a_{n-1}$ is a list of distinct elements of $\mathbb{F}_n$.

error correction

error correction

- in the following cryptosystems,

error correction

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- the Neiderreiter system uses GRS codes...

error correction

- in the following cryptosystems,
- the Neiderreiter system uses GRS codes...
- the McEliece system uses Goppa codes, a special sub-field sub-code family of GRS codes...